ACLT: Algebra, Categories, Logic in Topology
- Grothendieck's generalized topological spaces (toposes)

Steve Vickers
CS Theory Group
Birmingham

4. Toposes and geometric reasoning

How to "do generalized topology"

4. Toposes and geometric reasoning

Outline of course

1. Sheaves: Continuous set-valued maps

2. Theories and models: Categorical approach to many-sorted first-order theories.

3. Classifying categories: Maths generated by a generic model

4. Toposes and geometric reasoning: How to "do generalized topology"

Constructive!
No choice
No excluded middle

Classifying topos for T represents "space of models of T"

It is "geometric mathematics freely generated by generic model of T"

Map = geometric morphism
= result constructed geometrically from generic argument

Bundle = space constructed geometrically from generic base point
- fibrewise topology

Arithmetic universes for when you don't want to base everything on Set

Point-free topology

Point-set topology says:
1 - define collection of points as set
2 - define topology, using open subsets

Point-free topology describes points and opens in one single structure
- a geometric theory
- points are models
- opens are propositions

- sheaves are "derived sorts"

Sober spaces point-free

definition of sober

If X is a sober space, ΩX its frame of opens

points are determined by ΩX: they are the completely prime filters

topology is determined by ΩX:

open sets of completely prime filters are those of the form

$\{F \mid U \in F\}$ for some U

If Y also sober

continuous maps X -> Y are determined by the frames:

They are the frame homomorphisms ΩY -> ΩX

If f: X -> Y continuous, then inverse image f^{-1} is frame homomorphism

Other way round: note that completely prime filters of ΩX are frame homomorphisms ΩX -> Ω = frame of truth values.

Composing with a frame homomorphism ΩY -> ΩX

gives continuous map X -> Y

Locales

Any frame A can be treated as a point-free space.

From that point of view call it a locale

Continuous maps between locales are just frame homomorphisms backwards

As propositional geometric theory:
- A = signature. Write (a) for a as propositional symbol
- axioms

$$(a) \vdash (b) \quad \text{if } a \leq b$$
$$(a) \wedge (b) \vdash (a \wedge b)$$
$$\top \vdash (\top)$$
$$(\textstyle\bigvee_i a_i) \vdash \bigvee_i (a_i)$$

joins, finite meets in A become disjunctions, conjunctions in logic

models = completely prime filters of A
Lindenbaum algebra = A

Continuous maps are geometric morphisms

For propositional case:

presheaves with pasting

Theorem Let A, B be frames,
let Sh(A), Sh(B) be their toposes of sheaves.
Then there is a bijection between
- frame homomorphisms B -> A
- isomorphism classes of geometric morphisms Sh(A) -> Sh(B)

Proof idea
Elements of A (opens) correspond to subsheaves of 1.
If f: Sh(A) -> Sh(B) is a geometric morphism, then f* maps opens of B to opens of A, and gives a frame homomorphism B -> A.

Every sheaf is a colimit of opens, so f* is determined up to isomorphism by its action on opens.

Moreover, an arbitrary frame homomorphism gives rise to a geometric morphism.

Continuous maps are geometric morphisms

For propositional theories:

Geometric morphisms match continuous maps for locales
- which match continuous maps for sober spaces

For general theories:

Define continuous map to be geometric morphism (between classifying toposes)

Remember: geometric morphisms are equivalent to -
- functors in the opposite direction
- preserving finite limits and arbitrary lolimits

Reasoning in point-free logic

Box is classifying topos S[T]
Its internal mathematics is
- geometric mathematics
freely generated
by a (generic) model of T

Let M be a model of T ...

Reasoning here must be geometric
- finite limits, arbitrary colimits
- includes wide range of free algebras
- e.g. finite powerset
- *not* full powerset or exponentials
- it's predicative

To get f* to another topos E:
Once you know what M maps to (a model in E)
- the rest follows
- by preservation of colimits and finite limits

Reasoning in point-free logic

Let M be a model of T_1 ...

Geometric reasoning
- inside box

Then f(M) = ... is a model of T_2

Outside box

Get map (geometric morphism) f: S[T_1] -> S[T_2]

Geometric morphism transforms points

Idea

Classifying topos S[T_1] somehow "is" space of models of T.

Its points are the models of T - but where?

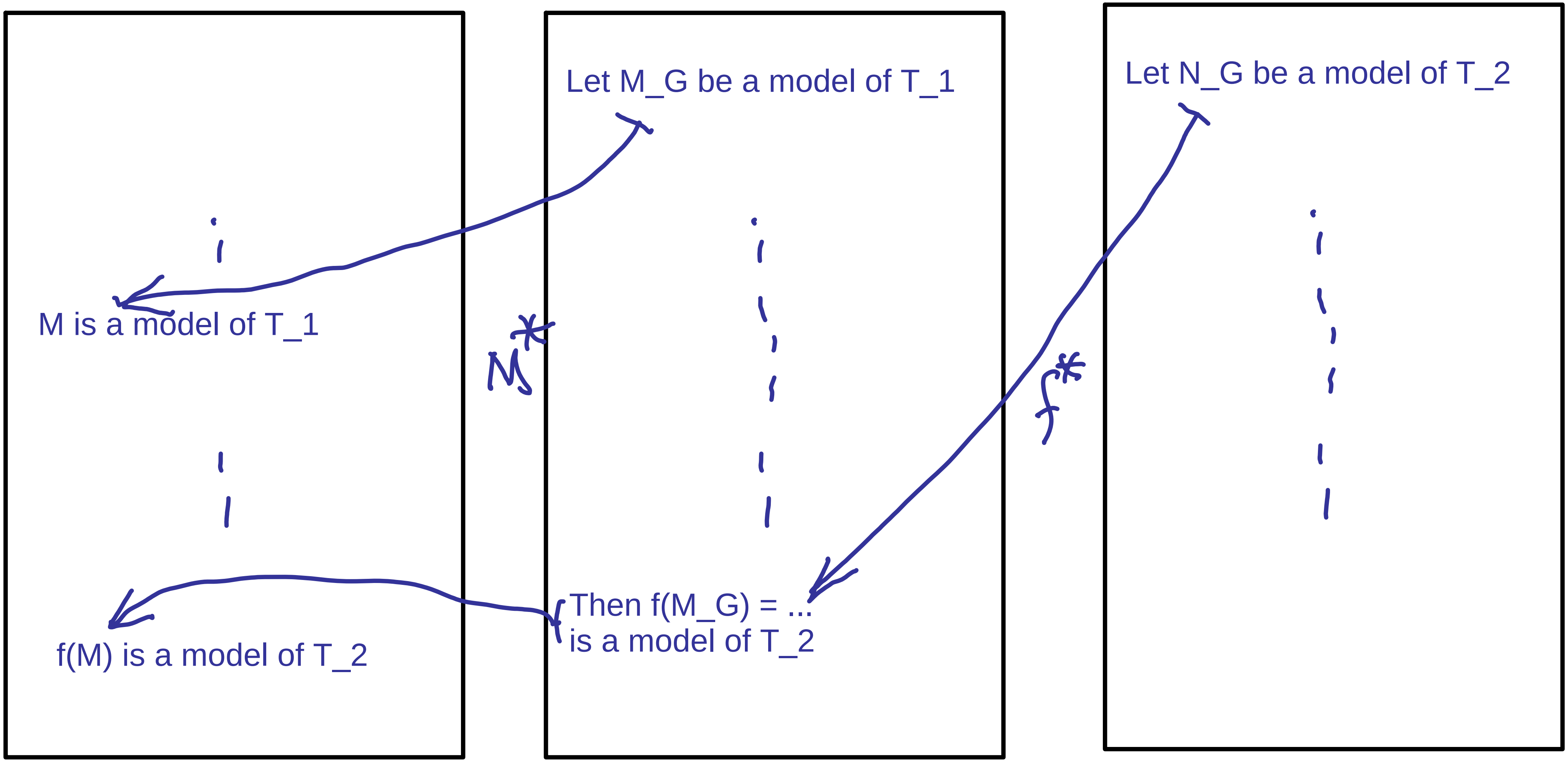
Model in E is equivalent to geometric morphism M: E -> S[T_1]

$$\mathcal{E} \xrightarrow{M} \mathcal{S}[T_1] \xrightarrow{f} \mathcal{S}[T_2]$$

Composing them gives model f(M) of T_2 in E

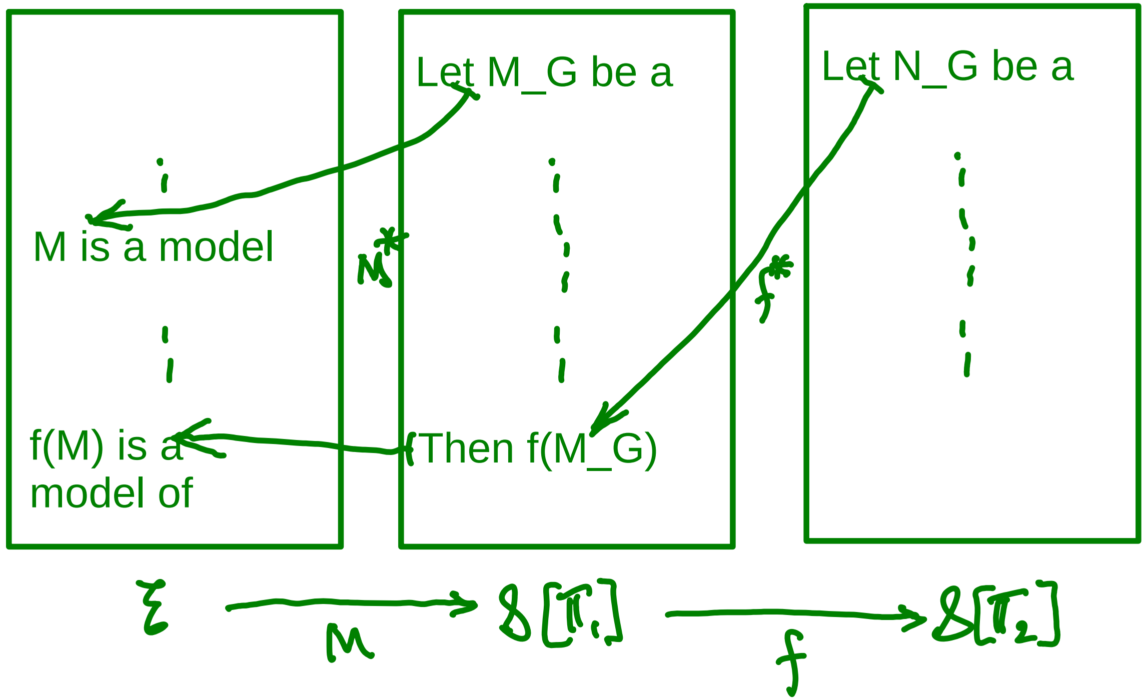
f transforms models of T_1 to models of T_2, in any E.

Reasoning in point-free logic



$$\Sigma \xrightarrow{M} \mathcal{S}[\mathbb{T}_1] \xrightarrow{f} \mathcal{S}[\mathbb{T}_2]$$

Role of geometricity



Construction of f(M_G) out of
M_G was geometric

Non-geometric constructions
(e.g. exponentials, powerobjects)
are also available in S[T_1]

They too could construct model of T_2 and give geometric morphism f.

But they wouldn't be preserved by M*
- they wouldn't construct M*(f(M_G)) out of M

If construction on generic model M_G is geometric,
then it is uniform
- same construction also applies to all specific models M.

To define a continuous map

- from space of T_1 models to space of T_2 models

1. Take as argument M a T_1 model

2. Construct a T_2 model f(M), geometrically

3. No continuity proof needed
- geometricity guarantees continuity

"geometricity *is* continuity"

Aspects of continuity

For ordinary spaces:

Continuous maps preserve specialization order

$x \sqsubseteq x' \equiv_{def} \mathcal{F}_x \subseteq \mathcal{F}_{x'}$

$\Rightarrow \mathcal{F}_{f(x)} \subseteq \mathcal{F}_{f(x')}$

For sober spaces:

Have all directed joins of points

Continuous maps preserve them

If $(x_i)_{i \in I}$ directed: then

$x = \bigsqcup_i x_i$ defined by

$\mathcal{F}_x = \bigcup_i \mathcal{F}_{x_i}$

$\mathcal{F}_{f(x)} = \bigcup_i \mathcal{F}_{f(x_i)}$

cf. Scott continuity

Aspects of continuity

For ordinary spaces:
Continuous maps preserve specialization order
For sober spaces:
Have all directed joins of points
Continuous maps preserve them

For generalized spaces

Specialization order becomes homomorphisms of models

Continuous maps (geometric morphisms) are functorial on points

$$\mathcal{E} \underset{M_2}{\overset{M_1}{\rightrightarrows}} \mathcal{S}[\mathbb{T}_1] \xrightarrow{f} \mathcal{S}[\mathbb{T}_2]$$

Homomorphism $M_1 \to M_2$

models

$\approx$ Natural transformation $M_1 \to M_2$

geometric morphisms

$\to$ Natural transformation $f \circ M_1 \to f \circ M_2$

$\approx$ Homomorphism $f(M_1) \to f(M_2)$

Aspects of continuity

For ordinary spaces:
Continuous maps preserve specialization order
For sober spaces:
Have all directed joins of points
Continuous maps preserve them

For generalized spaces

Instead of directed joins, consider filtered colimits

Defn A category C is filtered if any finite diagram in it has a cocone

1. C has at least one object

Empty diagram has a cocone

2. For any two objects i and j, there are morphisms out to a third k

$$\begin{tikzcd} i \arrow[dr] & \\ & k \\ j \arrow[ur] & \end{tikzcd}$$

3. For any two parallel morphisms, there is a third that composes equally with them

$$\begin{tikzcd} i \arrow[r, shift left] \arrow[r, shift right] & j \arrow[r] & k \end{tikzcd}$$

e.g. a poset is filtered iff it is directed

Filtered colimits

A filtered colimit is a colimit of a filtered diagram,
i.e. a functor from a filtered category

In Set: Suppose X: C -> Set a filtered diagram.

Then (theorem) its colimit is

$$\coprod_{i \in ob(\mathcal{C})} X_i \,/\, \sim$$

$x \in X_i,\ x' \in X_{i'}$

where $(i, x) \sim (i', x')$

if $\exists$
$$\begin{tikzcd} i \arrow[dr, "u"] & \\ & j \\ i' \arrow[ur, "u'"'] & \end{tikzcd}$$
such that $X(u)(x) = X(u')(x')$

Filtered colimits

Facts

Filtered colimits commute with finite limits

For a geometric theory T, filtered colimits of models can be found by taking filtered colimits of the carriers

For any two Grothendieck toposes, we have filtered colimits of geometric morphisms between them

Those filtered colimits are preserved by composition on either side

As point transformers, geometric morphisms preserve filtered colimits

Object classifier

Let O be theory with one sort and nothing else
Model = set

"set" here = object in a topos

S[O] is the object classifier, the "space of sets"

Map F: E -> S[O]
= (1) "continuous set-valued map on E"
= (2) object of E

definition of classifying topos

If E = Sh(X), this justifies "sheaf = continuous set-valued map"

If x is point of X, then F(x) = stalk at x (fibre of local homeomorphism)

Proof method Treat x as map x: 1 -> X, calculate x*(F).
It's an object of Sh(1) = Set

Objects of S[O]

Intuition Object of S[O] is
- continuous map from "space of sets" to itself

Continuity is at least functoriality and preservation of filtered colimits

- functor Set -> Set preserving filtered colimits

Every set is a filtered colimit of finite sets

Set is ind-completion of Fin

- functor Fin -> Set

Not a proof, but ...

Theorem S[O] is equivalent to [Fin, Set]

More generally
For any cartesian theory T, S[T] is equivalent to category of set-valued functors on category of finitely presented T-models

Reasoning in point-free topology: examples

$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

Dedekind sections, e.g. (L_x, R_x)

Let $x, y \in \mathbb{R}$

Then $x+y \in \mathbb{R}$ where

$$L_{x+y} = \{q+r \mid q \in L_x, r \in L_y\}$$

$$R_{x+y} = \{q+r \mid q \in R_x, r \in R_y\}$$

Why is real line R geometric?

1. Propositional theory

Propositional symbols for subbasic open intervals (q, ∝), (-∝, q)
  (q rational)
Axioms to express relations between these, e.g.

$$(q,\infty) \vdash \bigvee_{q<q'} (q',\infty)$$

infinite disjunction!

2. First order theory
- sorts N, Q
- structure and axioms to force them to be modelled as natural numbers and rationals
- predicates L(q:Q) and R(q:Q) for left and right parts of a Dedekind sections
- appropriate axioms

need infinite disjunctions to do this

Can show (1) and (2) are equivalent - mutually inverse maps between them.

Reasoning in point-free topology: examples

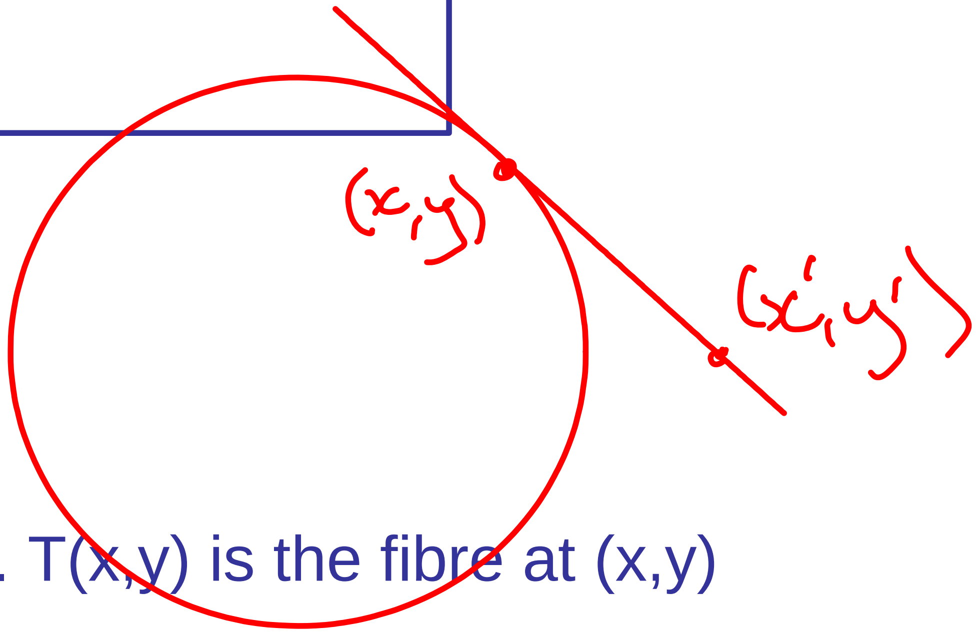
Let (x,y) be on the unit circle

$x^2 + y^2 = 1$

Then can define presentation for a subspace of RxR,
the points (x', y') satisfying
  xx' + yy' = 1

This construction is geometric

It's the tangent of the circle at (x,y)



Inside the box:
For each point (x,y), a space T(x,y)

Outside the box:
Defines the tangent bundle of the circle. T(x,y) is the fibre at (x,y)

Joyal and Tierney:
Internal point-free space = external bundle

fibrewise topology of bundles

Fibrewise topology

Let X be a space

Imagine topology "continuously parametrized by points x of X"
Hope to do topology as usual, but with parameters x everywhere

e.g. define spaces Y_x
bundle them together to make space Y with map p: Y -> X
  Y_x = fibre of p over x

- each fibre Y_x has given topology
- but what about topology of Y *across* the fibres?
- somehow comes from "continuity" of x |-> Y_x ???

Makes sense if -
- spaces are point-free
- construction of Y_x is geometric

James: "Fibrewise Topology"
- classical, point-set

Fibrewise topology

S[T1]

Let M_G be a point of T1 ...
:
:
Then F(M_G) is a space

geometric theory

Externally: get theory T2, models = pairs (M, N) where
- M a model of T1
- N a model of F(M)

Map p: S[T2] -> S[T1]
- (M,N) |-> M

Fibres

Can generalize to models in other toposes

Suppose M a model of T1 in Set = Sh(1)
Get map M: Set -> S[T1]

S[F(M)] = "fibre of p over M"

Get square -

$$
\begin{tikzcd}
\mathcal{S}[F(M)] \arrow[r] \arrow[d] & \mathcal{S}[\mathbb{T}_2] \arrow[d, "p"] \\
\mathbf{Set} \arrow[r, "M"] & \mathcal{S}[\mathbb{T}_1]
\end{tikzcd}
$$

Fact It's a (pseudo)pullback
in category Top of
Grothendieck toposes,
geometric morphisms

cf. pullback square for
ordinary fibres

$$
\begin{tikzcd}
p^{-1}\{x\} \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\
1 \arrow[r, "x"] & X
\end{tikzcd}
$$

Geometric morphisms: two views

f: S[T2] -> S[T1]

1. Map

argument y |-> result f(y)

2. Bundle

base point x |-> fibre f^{-1}{x}

Technicality when generalizing to elementary toposes:

for bundle view, f must be bounded

Localic bundle theorem (Joyal and Tierney)

Let E be any topos

Then there is a duality between:
- internal frames in E
- bundles p: F -> E that are localic

Technically:
every object of F is quotient of a subobject of some p*(X)

Hence:
internal locale maps correspond to external bundle morphisms

Note -
frames are not part of the geometric mathematics

Need powersets to construct them

Frame presentations (propositional geometric theories) are geometric

Geometric properties of bundles

Some topological properties C of spaces
- e.g. discreteness, compactness, local connectedness, ...
are preserved under (pseudo)pullback of bundles

Then say C is a geometric property

Say bundle p: Y -> X is fibrewise C iff it is internally C in Sh(X)

Then all its fibres are also C

e.g. discreteness

Discrete space = set (or object in topos)

Object X (in topos E)
- powerobject P(X) = frame for discrete space
- geometric theory T
  signature: propositional symbols s_x (x in X)
  axioms:

$$\top \vdash \bigvee_{x \in X} s_x$$

$$s_x \wedge s_y \vdash \bigvee \{\top \mid x = y\}$$

point = model of theory = singleton subset of X = element of X
open = formula = arbitrary subset of X (discrete topology)

Corresponding bundle over E is fibrewise discrete

Note: in this case bundle topos is equivalent to slice E/X.

Local homeomorphisms (Joyal and Tierney)

Let A be an internal frame in topos E, let p: F -> E be the bundle

Theorem The following are equivalent.
- p is fibrewise discrete (A is isomorphic to some P(X))
- p is open and so is Δ: F -> F x_E F

$$\begin{tikzcd} F \arrow[rr, "\Delta"] \arrow[dr, "p"'] & & F \times_E F \arrow[dl, "\pi_1"] \\ & E & \end{tikzcd}$$

cf. Lecture 1!
Use this as point-free definition of local homeomorphism

(But first have to define open maps.)

Fibrewise discrete = local homeomorphism

Point-set topology is inadequate!

With respect to base space X:

- space = bundle over X
- set = discrete space = local homeomorphism over X

"set of points" for a bundle p = approximation by a local homeomorphism

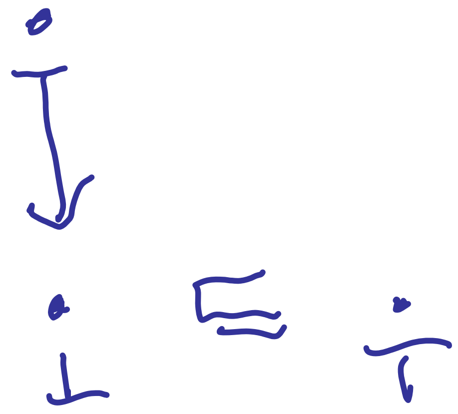
Sometimes no approximation is good enough.

$$\begin{tikzcd} \mathrm{pt}\left(\begin{smallmatrix} Y \\ \downarrow \\ X \end{smallmatrix}\right) \arrow[r] \arrow[dr] & Y \arrow[d, "p"] \\ & X \end{tikzcd}$$

For local homeomorphism,
specialization in base gives
map between fibres

pt(p) must have empty fibre over top
- so must also have empty fibre
over bottom

$$\begin{tikzcd} Y = 1 \arrow[d, "p"] \\ X = \$ \end{tikzcd}$$



Reasoning in point-free topology: examples

Spec: [BA] -> Spaces

Let B be a Boolean algebra

Then Spec (B) is point-free space of prime filters of B, presented by -

- generators $(b) \qquad (b \in B)$
- relations

$$(b_1 \wedge b_2) = (b_1) \wedge (b_2)$$

$$(1) = \top$$

$$(b_1 \vee b_2) = (b_1) \vee (b_2)$$

$$(0) = \bot$$

meet, join in B

logical conjunction, disjunction

Reasoning in point-free topology: examples

Spec : $[BA] \to$ Spaces

Let B be a Boolean algebra

Then Spec(B) is point-free space of prime filters of B, presented by

- generators (b) $(b \in B)$
- relations $(b_1 \wedge b_2) = (b_1) \wedge (b_2)$
  - $(1) = \top$
  - $(b_1 \vee b_2) = (b_1) \vee (b_2)$
  - $(0) = \bot$

meet, join in B

logical conjunction

disjunction

- B a pt of space of Boolean algebras
- internal point-free space
  = external bundle

$$\begin{tikzcd} {[BA + \text{prime filter}]} \arrow[d, "\text{forget prime filter}"] \\ {[BA]} \end{tikzcd}$$

Spec(B) is fibre over B

$$\begin{tikzcd} \mathrm{Spec}(B) \arrow[r] \arrow[d] & {[BA + \text{prime filter}]} \arrow[d] \\ \{\} \arrow[r, "B"'] & {[BA]} \end{tikzcd}$$

Geometricity => construction is uniform:

- single construction on generic B
- also applies to specific B's
- get those by pullback

pullback
= generalized fibre of generalized point

Localic hyperspaces (powerlocales)

Very useful geometrically

Point-free treatment of Vietoris topology
- split into two halves

Lower powerlocale P_L(X)
- point = "overt, weakly closed sublocale of X"
- specialization order = sublocale order

Think: closed subspace

Think: compact, and up-closed under specialization order

Upper powerlocale P_U(X)
- point = "compact, fitted sublocale of X"
- specialization order = opposite of sublocale order

Both work internally in geometric way
- giving fibrewise hyperspaces of bundles

Hyperspace applications - examples

Lower powerlocale P_L(X)
- point = "overt, weakly closed sublocale of X"
- specialization order = sublocale order

Upper powerlocale P_U(X)
- point = "compact, fitted sublocale of X"
- specialization order = opposite of sublocale order

X is compact iff
- there is a point K of P_U(X)
- such that

$$K \circ ! \sqsubseteq \mathrm{Id}_{P_U X}$$

$$\begin{tikzcd} 1 \arrow[r, shift right, "K"'] & P_U X \arrow[l, shift right, "!"'] \end{tikzcd}$$

K is biggest compact fitted sublocale,
and can show it must be the whole of X.

p: Y -> X is open iff
- there is a section L of (P_L)^X(Y)
- such that

$$\mathrm{Id}_{P_L^X(Y)} \sqsubseteq L \circ q$$

fibrewise lower powerlocale of p

$$\begin{tikzcd} Y \arrow[dr, "p"'] & & P_L^X(Y) \arrow[dl, shift right, "q"'] \\ & X \arrow[ur, shift right, "L"'] & \end{tikzcd}$$

p open iff ...

intuition:

$$\begin{tikzcd} Y \arrow[dr, "p"'] & & P_L^X(Y) \arrow[dl, shift right, "q"'] \\ & X \arrow[ur, shift right, "L"'] & \end{tikzcd}$$

For any Y: subbase of opens for $P_L Y$

$\Diamond V = \{A \mid A \cap V \neq \emptyset\}$ (V open in Y)

For $p : Y \to X$ - write Y_x for fibre over x

If V open in Y, $V \cap Y_x$ open in Y_x

$\Diamond(V \cap Y_x)$ open in $P_L(Y_x)$

$(\Diamond(V \cap Y_x))_{x \in X}$ open in $P_L^X(Y)$

Inequality $\mathrm{Id}_{P_L^X(Y)} \sqsubseteq L \circ q$ says $L(x) = Y_x$

$L^{-1}(\Diamond(V \cap Y_x))_{x \in X} = \{x \mid Y_x \in \Diamond(V \cap Y_x)\} = \{x \mid V \cap Y_x \neq \emptyset\}$

$=$ image $p(V)$

L continuous $\Rightarrow$ p open

Topos theory to do fibrewise topology of bundles

Programme:

Carry out topology in a way that is
- point-free (but can still use points!)
- geometric (reasoning must be constructive)

In scope of declaration "Let x be a point of X"
- x = generic point of X (in Sh(X))
- space = generic fibre of bundle over X
- geometric properties of space hold fibrewise

Hyperspaces are very useful
- internal hyperspace works fibrewise

Conclusions

Grothendieck's generalized spaces:

- can understand topology and continuity much more broadly than before
- sheaves are more important than opens
- sheaves provide a rich geometric mathematics for performing generic constructions on generic points

Even for ungeneralized spaces:
Topos theory -
constructive (geometric) point-free reasoning using sheaves over base

gives natural fibrewise topology of bundles
- topology parametrized by base point

Further reading

Topos theory -
Mac Lane and Moerdijk "Sheaves in geometry and logic"
Johnstone "Sketches of an elephant"
+ a readers' guide to those two -
Vickers "Locales and toposes as spaces"

Constructive reasoning for locales -
Joyal and Tierney "An extension to the Galois theory of Grothendieck"

Powerlocales -
Vickers - various papers; in particular
"The double powerlocale and exponentiation"