Clones (3&4)

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Galois connections

Let \( A, B \) be sets, \( R \subseteq A \times B \).
For any \( S \subseteq A \) and any \( T \subseteq B \) let

- \( S^u := \{ b \in B \mid \forall a \in S : aRb \} \)
- \( T^\ell := \{ a \in A \mid \forall b \in T : aRb \} \).

Then

- the maps \( T \mapsto T^\ell \) and \( S \mapsto S^u \) are \( \subseteq \)-antitone.
- the maps \( S \mapsto \bar{S} := S^{ul} \) and \( T \mapsto \bar{T} := T^{lu} \) are closure operators (\( S \subseteq \bar{S} = \bar{\bar{S}} \))
- \( S^{ulu} = S^u, T^{lu} = \ell. \)

Usually it is of interest to characterize the family of closed sets \( \{ S \mid S = \bar{S} \} \) and the closure operator “from below”.

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Galois connections, examples

Let $A$, $B$ be sets, $R \subseteq A \times B$.

$S^u := \{ b \in B \mid \forall a \in S : aRb \} = \bigcap_{a \in S} \{ b \in B \mid aRb \}$

$T^\ell := \{ a \in A \mid \forall b \in T : aRb \} = \bigcap_{b \in T} \{ a \in A \mid aRb \}$

Examples

- $A =$ vector space, $B =$ dual space = set of linear forms.
  $aRb \iff b(a) = 0$.
  $S \subseteq A \Rightarrow S^{u\ell} =$ linear hull of $S$.

- $A =$ all formulas, $B =$ all structures,
  $aRb \iff b \models a$ (the formula $a$ holds in the structure $b$).
  $\bar{S} =$ all consequences of $S = \{ a : S \models a \}$

- $A =$ operations on $X$, $B =$ relations, $f R \rho \iff f \triangleright \rho$.
  $S^{u\ell} = \langle S \rangle =$ clone generated by $S$. 
**k-ary clones, clones**

\( \mathcal{O}^{(k)} := \{ f \mid f : X^k \rightarrow X \} \). \( \mathcal{O}_X := \bigcup_{k=1}^{\infty} \mathcal{O}^{(k)}_X \).

**Definition (k-ary clone)**

A **k-ary clone** on \( X \) is a set \( T \subseteq \mathcal{O}^{(k)}_X \) which is closed under “composition” and contains the \( k \) projections.

**Definition (Clone)**

A **clone** on \( X \) is a set \( T \subseteq \mathcal{O}_X = \bigcup_{k=1}^{\infty} \mathcal{O}^{(k)}_X \) which is closed under “composition” and contains all projections.

**Definition (Composition)**

Let \( f \in \mathcal{O}^{(k)}, g_1, \ldots, g_k \in \mathcal{O}^{(m)}_X \).

\[ f(g_1, \ldots, g_k)(\vec{x}) := f(g_1(\vec{x}), \ldots, g_k(\vec{x})) \text{ for all } \vec{x} \in X^m. \]

If \( C \) is a clone, then \( C^{(k)} := C \cap \mathcal{O}^{(k)} \) is a \( k \)-ary clone, the **\( k \)-ary fragment** of \( C \).
Vector-valued operations

$C$ is a clone: $f, g_1, \ldots, g_k \in C \Rightarrow f(g_1, \ldots, g_k) \in C$.

We can view $(g_1, \ldots, g_k)$ as a single function $\vec{g} : X^m \to X^k$, and write $f \circ \vec{g}$ instead of $f(g_1, \ldots, g_k)$.

Definition

For any set $S \subseteq \emptyset_X$ let $\tilde{S}$ be the set of all operations $f : X^k \to X^n$ with the property that all “components” are in $S$:

$$\tilde{S} := \bigcup_{k,n} \{ f : X^k \to X^n \mid \forall i \in \{1, \ldots, n\} : \pi_i^n \circ f \in S \}$$

where $\pi_i^n : X^n \to X$ is the $i$-th projection function.

The set $S$ is a clone iff $\tilde{S}$ contains all projection functions and is closed under composition:

$$\forall g : X^m \to X^k \forall f : X^k \to X^n : (f, g \in \tilde{S} \Rightarrow f \circ g \in \tilde{S})$$
Examples of clones

- Every subset \( S \subseteq \emptyset_X \) will *generate* a clone \( \langle S \rangle \), the smallest clone containing \( S \).
- For any relation \( \rho \subseteq X^n \): \( \text{Pol}(\rho) := \{ f \in \emptyset_X^{(|f|)} \upharpoonright \rho \} \) is a clone.
- For any relation \( \rho \subseteq X^K \) (\( K \) infinite), \( \text{Pol}(\rho) \) is a clone.
- For any set \( R \) of relations, \( \text{POL}(R) := \bigcap_{\rho \in R} \text{Pol}(\rho) \) is a clone.
- \( \langle C \rangle = \text{POL}(\text{INV}(C)) \), where \( \text{INV}(C) := \bigcap_{f \in C} \text{Inv}(f) \), \( \text{Inv}(f) := \{ \rho \mid f \upharpoonright \rho \} \).
  (For infinite \( X \), need to allow infinitary relations; operations still have finite arity!)
The lattice of all clones on $X$

For finite $X$, $\mathcal{O}_X$ is countable.
For infinite $X$ of size $\kappa$, $\mathcal{O}_X$ has $2^\kappa$ elements.

Definition
For any nonempty set $X$ let $\text{Cl}(X)$ be the set of all clones on $X$.
($\text{Cl}(X)$ is a subset of the power set of $\mathcal{O}_X$.)

- $\text{Cl}(X)$ is a complete lattice. (meet = intersection, join = clone generated by union)
- $\text{Cl}(X)$ is Countable for $|X| = 2$.
  (Post’s lattice. wikipedia!)
- $\text{Cl}(X)$ is of size $|\mathbb{R}| = 2^{\aleph_0}$ for $X$ finite with $> 2$ elements.
- For infinite $X$ of size $\kappa$: $|\text{Cl}(X)| \leq 2^{2^\kappa}$.
  In fact: $= 2^{2^\kappa}$. (Later)
Minimal clones

Definition
We call a clone $M$ minimal if $J \varsubsetneq M$ ($J$ is the smallest clone, containing only the projections), but there is no clone $D$ with $J \varsubsetneq D \varsubsetneq M$.
The minimal clones are the atoms of the clone lattice.
An operation $m$ is minimal iff $\langle m \rangle$ is a minimal clone.
Instead of minimal clones we consider minimal operations.
If $m$ is minimal, then $\forall f \in \langle m \rangle \setminus J : m \in \langle f \rangle$.

- If $m$ is unary, then have $m \in \langle m^j \rangle$ for all $j$ except if $m^j = id$.
  Hence $j^2 = id$ (“retraction”), or $m$ is a permutation of prime order.
- If $m$ not essentially unary, then $m$ must be idempotent.
  \[ m(x, \ldots, x) = m. \]
Minimal operations, examples

▶ Every constant operation.
▶ Every permutation whose order is a prime number.
▶ The meet operation of any meet-semilattice.
▶ The median operation in any linear order.
▶ ... (many more. Some necessary conditions known, but no explicit criterion.)

Fact
If $X$ is finite, then there are finitely many minimal operations. Every clone $\neq J$ contains a minimal clone.

(This is not true for infinite sets. Let $s : \mathbb{Z} \to \mathbb{Z}$ be defined by $s(x) = x + 1$, then every non-projection in $\langle s \rangle$ is of the form $s^j$ ($j \in \{1, 2, \ldots \}$, and none of them is minimal, as $\langle s^{2j} \rangle \subsetneq \langle s^j \rangle$.)
Complete sets

Theorem

For every $X$: $\langle \mathcal{O}_X^{(2)} \rangle = \mathcal{O}_X$.

Proof for infinite $X$.

- Let $p_2 : X^2 \rightarrow X$ be a bijection.
- Find bijections $p_j : X^j \rightarrow X$ for $j = 3, 4, \ldots$, with $p_j \in \langle \mathcal{O}^{(2)} \rangle$.
  For example, $p_3(x, y, z) := p_2(x, p_2(y, z))$.
- For every $f : X^k \rightarrow X$, let $\hat{f} := f \circ p_k^{-1}$.
  So $f(\vec{x}) = \hat{f}(p_k(\vec{x}))$ for all $\vec{x} \in X^k$. As $\hat{f}$ is unary, $\hat{f} \in \langle \mathcal{O}^{(2)} \rangle$.
- From $\hat{f} \in \langle \mathcal{O}^{(2)} \rangle$ and $p_k \in \langle \mathcal{O}^{(2)} \rangle$ conclude $f \in \langle \mathcal{O}^{(2)} \rangle$. 
Complete sets

For every $X$: $\langle O_X^{(2)} \rangle = O_X$.

Proof for finite $X$ ("Lagrange interpolation").

Let $(X, +, \cdot, 0, 1)$ be a finite lattice with smallest element 0 and greatest element 1. So $x + 0 = 0 + x = x = 1 \cdot x$ for all $x$.

- For each $a \in X$ let $\chi_a : X \to X$ be the characteristic function of the set $\{a\}$. So $\chi_a \in O^{(1)} \subseteq \langle O^{(2)} \rangle$.
- For each $\bar{a} \in X^k$ let $\chi_{\bar{a}} : X^k \to X$ be the characteristic function of $\{\bar{a}\}$: $\chi_{\bar{a}} = \prod_i \chi_{a_i}(x_i)$. So $\chi_{\bar{a}} \in \langle O^{(2)} \rangle$.
- For any $b \in X$ let $c_b \in O^{(1)}$ be constant with value $b$.
- Every operation $f \in O^{(k)}$ can now be written as $f = \sum_{\bar{a} \in X^k} (\chi_{\bar{a}} \cdot c_{f(\bar{a})})$. So $f \in \langle O^{(2)} \rangle$.

(Remark: This proof also works for strongly amorphous sets.)
Precomplete clones

Definition
A clone $C \subseteq \emptyset_X$ is “precomplete” (or “maximal”) if $C \neq \emptyset_X$, but there is no clone $D$ satisfying $C \subsetneq D \subsetneq \emptyset_X$.

Theorem
*For any clone $C \subsetneq \emptyset_X$ there is a precomplete clone $C'$ with $C \subseteq C'$.*

(Remark: Not true for infinite sets! At least if the continuum hypothesis holds.)
Post’s lattice

The lattice of all clones on a 2-element set is countably infinite.

It has 5 coatoms ("precomplete" clones) and 7 atoms.
Precomplete clones, example 1

Let \( \rho \) be a nontrivial unary relation, i.e. \( \emptyset \subsetneq \rho \subsetneq X \). Then \( \text{Pol}(\rho) \) is the set of all operations \( f \) such that \( \rho \) is a subalgebra of \((X, f)\). This clone is precomplete.

Proof.
Let \( g : X^k \rightarrow X, g \notin \text{Pol}(\rho) \). Let \( C := \langle \text{Pol}(\rho) \cup \{g\} \rangle \). We show \( C = \emptyset_X \). Sufficient: \( C \supseteq \emptyset_X^{(2)} \).

For \( v \in X \), let \( c_v \) be the constant function with value \( v \).
There are \( \bar{a} = (a_1, \ldots, a_k) \in \rho^k \), \( b \notin \rho \) with \( g(\bar{a}) = b \),
So \( c_b = g(c_{a_1}, \ldots, c_{a_k}) \) is in \( C \).

For \( f \in \emptyset_X^{(2)} \) define \( \hat{f}(x_1, x_2, y) := \begin{cases} x_1 & \text{if } y \in \rho \\ f(x_1, x_2) & \text{if } y \notin \rho \end{cases} \). So \( \hat{f} \in C \).

Now \( f = \hat{f}(\pi_1^2, \pi_2^2, c_b) \), i.e., \( f(x_1, x_2) = \hat{f}(x_1, x_2, b) \). So \( f \in C \).
Precomplete clones, example 2

\(\sim\) a nontrivial equivalence relation \(\Rightarrow\) \(\text{Pol}(\sim)\) is precomplete.

Proof.
For \(\vec{a}, \vec{b} \in X^k\) write \(\vec{a} \sim \vec{b}\) iff \(\forall i\ a_i \sim b_i\). This is an equivalence relation on \(X^k\).
Let \(g : X^k \to X, g \notin \text{Pol}(\sim)\). Let \(C := \langle \text{Pol}(\sim) \cup \{g\} \rangle\). We have to show \(C = \emptyset_X\). Sufficient: \(C \supseteq \emptyset_X^{(2)}\).
There is \(k\) and \(\vec{a} \sim \vec{b} \in X^k\) with \(1 := g(\vec{a}) \not\sim g(\vec{b}) =: 0\).
We claim that for each \(p \in X^2\) there is a function \(\chi_p : X^2 \to X\) which maps \(p\) to \(1\), everything else to \(0 \not\sim 1\).
For each \(p \in X^2\) let \(h_p : X^2 \to X^k\) be defined by \(h_p(p) = \vec{a}, h_p(x) = \vec{b}\) otherwise. Clearly \(h_p \in \text{Pol}(\sim)\). So \(\chi_p := g \circ h_p \in C\).
(continued on next page)
Proof that Pol(∼) is precomplete, continued.

We started with a clone \( C \supseteq \text{Pol}(\sim) \).

For each \( p \in X^2 \) we have found \( \chi_p \in C \), \( \chi_p : X^2 \to X \) with \( \chi_p(p) = 1 \), \( \chi_p(x) = 0 \) for \( x \neq p \). (And \( 0 \not\sim 1 \))

Define \( \chi : X^2 \to X^{\mid X \mid^2} \) by \( \chi(\vec{x}) = (\chi_p(x) : p \in X^2) \). So \( \chi \in \tilde{C} \).

Let \( f \in \mathcal{O}(\chi)^{(2)} \) be arbitrary. We will show \( f \in C \).

Define \( \hat{f} : X^{2+\mid X \mid^2} \to X \) as follows:

\[ \begin{align*}
\text{\( \hat{f} \) is constant on each \( \sim \)-class. (So \( \hat{f} \in \text{Pol}(\sim) \subseteq C \))} \\
\text{\( \hat{f}(\vec{x}, \chi(\vec{x})) = f(\vec{x}) \).}
\end{align*} \]

This two requirements are compatible, as \( \vec{x} \neq \vec{x}' \) implies that \( \chi(\vec{x}) \not\sim \chi(\vec{x}') \).

Clearly \( f(\vec{x}) = \hat{f}(\vec{x}, \chi(\vec{x})) \). So \( f \in C \).
Precomplete clones, example 3

Definition
Let $r : X \rightarrow X$, $f : X^k \rightarrow X$. We say that $f$ commutes with $r$ if:

$$\forall x_1, \ldots, x_k \in X : f( r(x_1), \ldots, r(x_k) ) = r( f(x_1, \ldots, x_k) )$$

Writing $r^\bullet$ for the relation $\{(x, r(x)) \mid x \in X\}$, $f$ commutes with $r$ iff $f \triangleright r^\bullet$. (We may write $f \triangleright r$ instead of $f \triangleright r^\bullet$)

Clearly $f \triangleright r \Rightarrow f \triangleright r^j$ for all $j$. Hence e.g. Pol$(r) \subseteq$ Pol$(r^2)$. But if $r$ is a permutation of order $p$, then Pol$(r) =$ Pol$(r^j)$ whenever $p$ does not divide $j$.

Theorem
Assume that $r : X \rightarrow X$ is a permutation and all cycles have the same prime length. Then Pol$(r)$ is precomplete.
Precomplete clones, examples 4,5

- “monotone”: Let \( \rho \subseteq X \times X \) be a partial order with smallest and greatest element. \( \text{Pol}(\rho) \) is the set of all pointwise monotone operations.

- “affine” Assume \( |X| = p^m \), so wlog \( X \) is a finite field \( X = GF(p^m) \).
  Let \( \rho = \{(a, b, c, d) \in X^4 \mid a + b = c + d\} \). Then \( \text{Pol}(\rho) \) is the set of all operations \( f \) of the form

\[
f(x_1, \ldots, x_k) = a_0 + \sum_{i=1}^{k} \sum_{j=0}^{m-1} x_i^p j
\]

All these clones are precomplete.
Post’s lattice, again

The 5 precomplete clones in $Cl(\{0, 1\})$:

- operations preserving $\{0\}$.
- operations preserving $\{1\}$.
- monotone operations
- “commuting”:
  $f(\neg x) = \neg f(x)$.
- affine operations
Rosenberg’s list

Theorem

Let $X = \{1, \ldots, k\}$. Then there is an explicit finite list of relations $\rho_1, \ldots, \rho_m$ such that every precomplete clone on $X$ is one of $\text{Pol}(\rho_1), \ldots, \text{Pol}(\rho_m)$.

The list includes

- all “central relations” (generalisations of $\rho \subsetneq X$)
- all nontrivial equivalence relations ($\forall$ if $|X| = 2$)
- all prime permutations
- All bounded partial orders
- affine relations (only if $|X| = p^n$)
- (others. more complicated but still explicit)
Rosenberg’s list

**Theorem**

Let \( X = \{1, \ldots, k\} \). Then there is an explicit finite list of relations \( \rho_1, \ldots, \rho_m \) such that every precomplete clone on \( X \) is one of \( \text{Pol}(\rho_1), \ldots, \text{Pol}(\rho_m) \).

**Completeness criterion** \( \langle S \rangle \neq \emptyset_X \) iff there is some \( \rho_i \) from the list with \( \forall f \in S : f \uparrow \rho_i \).
A complicated interval in the clone lattice

Definition
Let $C_{idem}$ be the clone of all idempotent operations:
$$f(x, \ldots, x) = x. \text{ (Assume } |X| \geq 3.)$$
Find all clones between $C_{idem}$ and $\emptyset_X$!

Example
Let $Y \subseteq X$. Then $C_{idem}|_Y := \{ f \mid \forall x \in Y : f(x, \ldots, x) = x \}$ is a clone $\supseteq C_{idem}$.

Theorem
Every clone between $C_{idem}$ and $\emptyset_X$ is of the form $C_{idem}|_Y$.
Hence: the interval $[C_{idem}, \emptyset_X]$ is (anti-)isomorphic to the power set of $X$.
(Precomplete clones correspond to singletons, $\emptyset_X$ to $\emptyset$.)
Let $C$ be a clone containing all idempotent operations $f(x, \ldots, x) = x$. We want to find $Y$ such that

$$C = C_{\text{idem}} \upharpoonright Y = \{ f \mid \forall y \in Y : f(y, \ldots, y) = y \}.$$ 

- \textbf{fix}(f) := \{ a \in X \mid f(a, \ldots, a) = a \}, \textbf{nix}(f) := X \setminus \text{fix}(f).
- Let $R := \{ \text{nix}(f) \mid f \in C \}$.
- $R$ is downward closed.
- $R$ is upward directed, hence an ideal.
- Let $Z$ be the largest element of $R$, $Y := X \setminus Z$.
- So $C \subseteq C_{\text{idem}} \upharpoonright Y$.
- If $\text{nix}(f) \subseteq \text{nix}(g)$ and $g \in C$, then $f \in C$.
- Hence $C = C_{\text{idem}} \upharpoonright Y$. 

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A complicated interval, continued

\( C_{\text{idem}} = \) the clone of all idempotent operations: \( f(x, \ldots, x) = x \).

**Theorem (X finite)**

*Every clone between \( C_{\text{idem}} \) and \( \emptyset_X \) is of the form \( C_{\text{idem}} \upharpoonright Y \).*

For infinite \( X \):

**Definition**

For every filter \( \mathcal{F} \) on \( X \), let

\[
  C_{\mathcal{F}} := \bigcup_{Y \in \mathcal{F}} C_{\text{idem}} \upharpoonright Y = \{ f \mid \exists Y \in \mathcal{F} \; \forall y \in Y \; f(y, \ldots, y) = y \}
\]

Each \( C_{\mathcal{F}} \) is a clone above \( C_{\text{idem}} \).
A complicated interval, conclusion

\[ C_{\mathcal{F}} := \{ f \mid \exists Y \in \mathcal{F} \ \forall y \in Y \ f(y, \ldots, y) = y \} \]

**Theorem**

Let \( X \) be any set. Then the map \( \mathcal{F} \mapsto C_{\mathcal{F}} \) is an order-preserving bijection between the filters on \( X \) and the clones above \( C_{\mathcal{F}} \).

Ultrafilters correspond to precomplete clones in this interval, and the improper filter corresponds to \( \emptyset_X \).

(For finite sets, all filters are principal.)

Translation to topology: the interval \( [C_{\text{idem}}, \emptyset_X] \) is anti-isomorphic to the family of closed sets of \( \beta X \), the Čech-Stone compactification of the discrete space \( X \). (Precomplete clones correspond to points, \( \emptyset_X \) to \( \emptyset \).)
Another complicated interval

Let $X$ be infinite. We will find “very many” clones with trivial unary fragment, i.e., below $C_{idem}$, the clone of all idempotent operations. (Unfortunately: no complete classification.)

In fact all our operations will be “conservative”:

$$f(x_1, \ldots, x_k) \in \{x_1, \ldots, x_k\}.$$  

- Let $(A_i : i \in I)$ be a family of sufficiently independent sets. 
  (In particular: we demand that for any finite $I_0 \subseteq I$ and any $j \in I \setminus I_0$ the set $(\bigcup_{i \in I_0} A_i) \cap (X \setminus A_j)$ contains at least 2 elements. It is possible to find such a family with $2^{|X|}$ elements, in particular: an uncountable such family.)

- Fix a linear order $\leq_i$ on $A_i$, with minimum operation $\wedge_i$.

- Extend $\wedge_i$ to $X$ by requiring $x \wedge_i y = x$ outside $A_i$.

- For any $I' \subseteq I$ let $C_{I'} := \langle\{\wedge_i \mid i \in I'\}\rangle$. Then all $C_{I'}$ are distinct. (Note: the numbers of such clones $= 2^{|X|}$.)
Local clones

Let $X$ be infinite. A clone $C$ is local if each fragment $C \cap \emptyset_X^k$ is closed in the product topology (pointwise convergence) on $X^{X^k}$ (with discrete $X$). Equivalently: If there is a set $R$ of relations of finite arity such that $C = \text{POL}(R)$.

The lattice of local clones has only $2^{|X|}$ elements; the lattice of all clones: $2^{2^{|X|}}$.

Example:

On a finite set with $k$ elements, the interval $[\emptyset_X^{(1)}, \emptyset_X]$ has $k + 1$ elements.

On any infinite set $X$, the interval $[\emptyset_X^{(1)}, \emptyset_X]$ in the lattice of all clones has at least $2^{2^{|\mathbb{N}|}}$ elements.

On any infinite set $X$, the interval $[\emptyset_X^{(1)}, \emptyset_X]$ in the lattice of local clones has at only countably many elements.
Bonus round: non-AC

We used $X \times X \approx X$ to show that $\langle \emptyset_X^{(2)} \rangle = \emptyset_X$ (for infinite sets $X$). But $X \times X \approx X$ uses the axiom of choice (and in fact $\forall X$ infinite : $X \times X \approx X$ is equivalent to AC). Was that necessary?
Yes, probably.


Let $(M, R_3)$ be the “random 3-uniform hypergraph”. That is, $R_3$ is a totally symmetric totally irreflexive relation which is “as random as possible”. For example: For all (reasonable) finite sets $\{a_1, b_1, \ldots, a_k, b_k, c_1, d_1, \ldots, c_n, d_n\} \subseteq M$ there is some $e \in M$ with $R(a_i, b_i, e)$ for all $i$, and $\neg R(c_j, d_j, e)$ for all $j$.
(Technically: the Fraïssé limit of all finite 3-uniform hypergraphs.)
non-AC, continued

Continuation of the proof.
Let \((M, R_3)\) be the “random 3-uniform hypergraph”. \((M\) countable, \(R_3 \subseteq M^3\) is “random” or “generic”.)
Let \(f_1, \ldots, f_m\) be first order definable binary operations, say definable from \(m_1, \ldots, m_k\) in the structure \((M, R)\). Then the set \(X \times X\) can be partitioned into finitely many sets according to the “type” a pair \((x, y)\) can have over \(m_1, \ldots, m_k\). On each type each operation \(f_i\) must be either constant or a projection, so the same is true for any element of \(\langle f_1, \ldots, f_k \rangle\). But the function \(\chi_R\) is neither a projection or a constant on any type. So we have found a definable ternary function not in the clone generated by the definable binary functions.
We have found a definable ternary function on \((M, R)\), definable from \(R\), but not in the clone generated by the definable binary functions.

Now construct a model of \(ZF+\neg\text{AC}\) in which all operations on \(M\) are definable from \(R\) and finitely many parameters. In this model, all binary operations are trivial on a large set, but not all ternary operations.
The clone lattice on \( \{0, 1\} \) is well understood. (But nontrivial.)

\( Cl(X) \) for larger finite sets \( X \): many fragments are explicitly known (certain intervals, coatoms, …), others only partially (atoms), or only for very small sets (say, \( |X| \leq 4, 5 \)).

To analyse \( k \)-ary operations, it is often helpful to consider \( k + 1 \)-ary operations. (Or \( 2k \)-ary. or \( (k + |X|^2) \)-ary, etc.)

Many open questions.

For infinite \( X \): set theory kicks in. Local clones more interesting than all clones?
Thank you for your attention! and for your questions! . . . and for your corrections!!