Arithmetic universes as generalized point-free spaces

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* Grothendieck: "A topos is a generalized topological space"
* ... it's represented by its category of sheaves
* but that depends on choice of base "category of sets"
* Joyal's arithmetic universes (AUs) for base-independence

"Sketches for arithmetic universes" (arXiv:1608.01559)
"Arithmetic universes and classifying toposes" (arXiv:1701.04611)

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Overall story

Open = continuous map valued in truth values
- Theorem: open = map to Sierpinski space $

Sheaf = continuous set-valued map
- no theorem here - "space of sets" not defined in standard topology
- motivates definition of local homeomorphism
- each fibre is discrete
- somehow, fibres vary continuously with base point

Can define topology by defining sheaves
- opens are the subsheaves of 1

But why would you do that?
- much more complicated than defining the opens

Generalized spaces (Grothendieck toposes)

But why would you do that?
- much more complicated than defining the opens

Grothendieck discovered generalized spaces
- there are not enough opens
- you have to use the sheaves
- e.g. spaces of sets, or rings, of local rings
- set-theoretically - can be proper classes
- generalized topologically:
- specialization order becomes specialization *morphisms*
- continuous maps must be *at least* functorial and preserve filtered colimits
- cf. Scott continuity

Outline

Point-free "space" = space of models of a geometric theory
- geometric maths = colimits + finite limits
- constructive
- includes free algebras, finite powersets
- but not exponentials, full powersets
- only a fragment of elementary topos structure
- fragment preserved by inverse image functors

cf. unions, finite intersections of opens

Space represented by classifying topos
= geometric maths generated by a generic point (model)

"continuity = geometricity"
- a construction is continuous if can be performed in geometric maths
- continuous map between toposes = geometric morphism
- geometrically constructed space = bundle, point |-> fibre
- "fibrewise topology of bundles"

Outline of tutorials

1. Sheaves: Continuous set-valued maps

2. Theories and models: Categorical approach to many-sorted first-order theories.

3. Classifying categories: Maths generated by a generic model

4. Toposes and geometric reasoning: How to "do generalized topology".

1. Sheaves

Local homeomorphism viewed as continuous map base point |-> fibre (stalk)

Alternative definition via presheaves

Idea: sheaf theory = set-theory "parametrized by base point"

Constructions that work fibrewise
- finite limits, arbitrary colimits
- cf. finite intersections, arbitrary unions for opens
- preserved by pullback

Interaction with specialization order

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2. Theories and models

(First order, many sorted)

Theory = signature + axioms
Context = finite set of free variables
Axiom = sequent

Models in Set
- and in other categories

Homomorphisms between models

Geometric theories

Propositional geometric theory => topological space of models.

Generalize to predicate theories?

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Describe so can be easily generalized from Set to any category with suitable structure

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Let M be a model of T ...

⋮

3. Classifying categories

Geometric theories may be incomplete
- not enough models in Set
- category of models in Set doesn't fully describe theory

generalizes Lindenbaum algebra

Classifying category - e.g. Lawvere theory
= stuff freely generated by generic model
- there's a universal characterization of what this means

For finitary logics, can use universal algebra
- theory presents category (of appropriate kind) by generators and relations

For geometric logic, classifying topos is constructed by more ad hoc methods.

4. Toposes and geometric reasoning

Outline of course

1. Sheaves: Continuous set-valued maps

2. Theories and models: Categorical approach to many-sorted first-order theories.

3. Classifying categories: Maths generated by a generic model

4. Toposes and geometric reasoning: How to "do generalized topology"

Constructive!
No choice
No excluded middle

Classifying topos for T represents "space of models of T"

It is "geometric mathematics freely generated by generic model of T"

Map = geometric morphism
= result constructed geometrically from generic argument

Bundle = space constructed geometrically from generic base point
- fibrewise topology

Arithmetic universes for when you don't want to base everything on Set

Universal property of classifying topos Set[T]

1. Set[T] has a distinguished "generic" model M of T.

2. For any Grothendieck topos E,
and for any model N of T in E,
there is a unique (up to isomorphism) functor f*: Set[T] -> E
that preserves finite limits and arbitrary colimits
and takes M to N.

Same idea as for frames

f* preserves arbitrary colimits
- can deduce it has right adjoint

These give a geometric morphism f: E -> Set[T]
- topos analogue of continuous map

More carefully: categorical equivalence between -
- category of T-models in E
- category of geometric morphisms E -> Set[T]

Reasoning in point-free logic

Box is classifying topos Set[T]
Its internal mathematics is
- geometric mathematics
freely generated
by a (generic) model of T

Let M be a model of T ...

Reasoning here must be geometric
- finite limits, arbitrary colimits
- includes wide range of free algebras
- e.g. finite powerset
- *not* full powerset or exponentials
- it's predicative

To get f* to another topos E:
Once you know what M maps to (a model in E)
- the rest follows
- by preservation of colimits and finite limits

Reasoning in point-free logic

Let M be a model of T_1 ...

Geometric reasoning
- inside box

Then f(M) = ... is a model of T_2

Outside box

Get map (geometric morphism) f: Set[T_1] -> Set[T_2]

Reasoning in point-free topology: examples

$+ : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$

Dedekind sections, e.g. (L_x, R_x)

Let $x, y \in \mathbb{R}$

Then $x + y \in \mathbb{R}$ where

$$L_{x+y} = \{q + r \mid q \in L_x, r \in L_y\}$$

$$R_{x+y} = \{q + r \mid q \in R_x, r \in R_y\}$$

Fibrewise topology

S[T1]

Let M_G be a point of T1 ...

:

:

geometric theory

Then F(M_G) is a space

Externally: get theory T2, models = pairs (M, N) where
- M a model of T1
- N a model of F(M)

Map p: Set[T2] -> Set[T1]
- (M,N) |-> M

Think of p as bundle, base point M |-> fibre F(M)

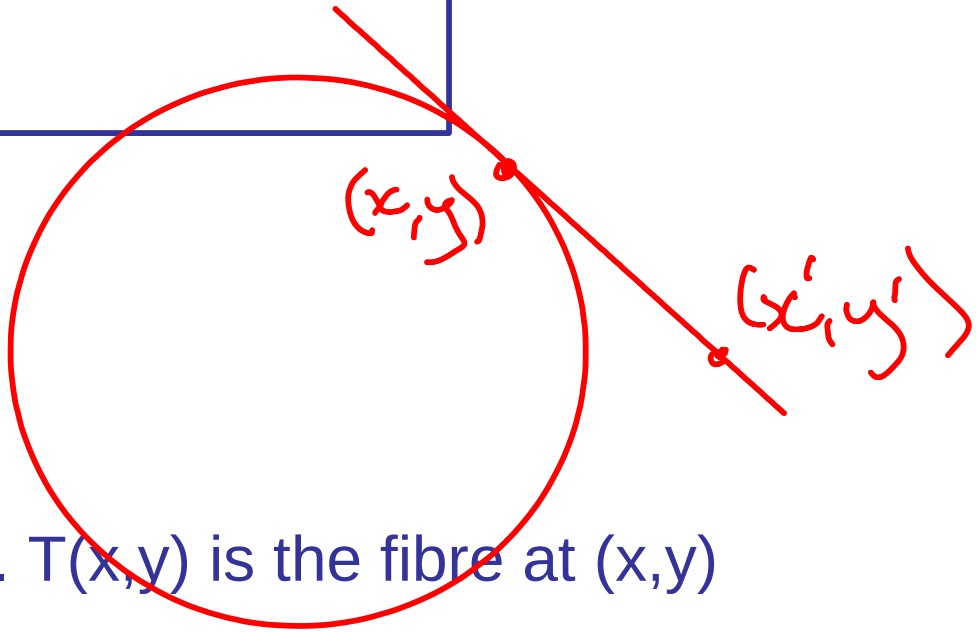
Reasoning in point-free topology: examples

Let (x,y) be on the unit circle $x^2 + y^2 = 1$

Then can define presentation for a subspace of RxR,
the points (x', y') satisfying
  xx' + yy' = 1

This construction is geometric

It's the tangent of the circle at (x,y)



Inside the box:
For each point (x,y), a space T(x,y)

Outside the box:
Defines the tangent bundle of the circle. T(x,y) is the fibre at (x,y)

Fourman & Scott; Joyal & Tierney:
Internal point-free space = external bundle

fibrewise topology of bundles

Example: "space of sets" (object classifier)

Theory $\mathbb{O}$ one sort, nothing else.

Classifying topos $\mathbf{Set}[\mathbb{O}] = [\mathrm{Fin}, \mathbf{Set}]$

Conceptually object = continuous map {sets} -> {sets}
Continuity is (at least) functorial + preserves filtered colimits
Hence functor {finite sets} -> {sets}

Generic model is the subcategory inclusion Inc: Fin -> Set

Example: "space of pointed sets"

Theory $\emptyset, pt$ one sort X, one constant x: 1 -> X.

Classifying topos $\mathbf{Set}[\emptyset, pt] \simeq [\mathrm{Fin}, \mathrm{Set}]/\mathrm{Inc}$

In slice category: 1 becomes Inc, Inc becomes Inc x Inc

Generic model is Inc with

$$\begin{tikzcd}
\mathrm{Inc} \arrow[rr, "\Delta"] \arrow[dr, "\mathrm{id}"'] & & \mathrm{Inc} \times \mathrm{Inc} \arrow[dl, "\pi_2"] \\
& \mathrm{Inc} &
\end{tikzcd}$$

1 in slice

Inc in slice

Generic local homeomorphism

$$
\begin{tikzcd}
\mathrm{Set}[\emptyset, \mathrm{pt}] \arrow[d, "p"'] & \text{"space of pointed sets"} \arrow[d, "\text{forget point}"] \\
\mathrm{Set}[\emptyset] & \text{"space of sets"}
\end{tikzcd}
$$

p is a local homeomorphism

Over each base point (set) X, fibre is discrete space for X

Every other local homeomorphism is a pullback of p

Suppose you don't like Set?

the base topos

Replace with your favourite elementary topos S.
Needs nno N.

$\mathcal{S}$

Fin becomes internal category in S.

Finite functions
f: m -> n

$$\mathrm{Fin}_1 \underset{\mathrm{cod}}{\overset{\mathrm{dom}}{\rightrightarrows}} \mathrm{Fin}_0 = \mathbb{N}$$

n = {0, ..., n-1}

Classifying topos becomes
- category of internal diagrams on Fin

$$\mathcal{S}[\mathbb{O}] = [\mathrm{Fin}, \mathcal{S}]$$

(f: m -> n, x in X(m))

$$\begin{tikzcd} \mathrm{dom}^* X \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & X \arrow[d] \\ \mathrm{Fin}_1 \arrow[r, shift left, "\mathrm{dom}"] \arrow[r, shift right, "\mathrm{cod}"'] & \mathbb{N} \end{tikzcd}$$

X(n) = fibre over n

X(f)(x) in X(n)

Suppose you don't like impredicative toposes?

Be patient!

Other classifier is slice, as before.

Generic local homeomorphism over $\mathcal{S}$

$$\begin{tikzcd} \mathcal{S}\ \cancel{\mathbf{Set}}[\emptyset, pt] \arrow[d, "p"'] & \text{"space of pointed sets"} \arrow[d, "\text{forget point}"] \\ \mathcal{S}\ \cancel{\mathbf{Set}}[\emptyset] & \text{"space of sets"} \end{tikzcd}$$

p is a local homeomorphism

Over each base point (set) X, fibre is discrete space for X

Every other local homeomorphism between toposes bounded over S is a pullback of p

Roles of S

Infinities are extrinsic to logic
- supplied by S

(1) Supply infinities for infinite disjunctions:
get theories T geometric over S.

(2) Classifying topos built over S: geometric morphism $\mathcal{S}[\mathbb{T}] \to \mathcal{S}$

Suppose T has disjunctions all countable

It's geometric over any S with nno.

But different choices of S give different classifying toposes.

Idea: use finitary logic with type theory that provides nno
- replace countable disjunctions by existential quantification over countable types
- they become intrinsic to logic
- a single calculation with that logic gives results valid over any suitable S

cf. suggestion in Vickers "Topical categories of domains" (1995)

Arithmetic universes instead of Grothendieck toposes

Pretopos -

finite limits
coequalizers of equivalence relations
finite coproducts

+ all well behaved

&

+ set-indexed coproducts
+ smallness conditions

Giraud's theorem

Grothendieck toposes
bounded S-toposes

extrinsic infinities from S

+ parametrized list objects

$$1 \xrightarrow{\varepsilon} \mathrm{List}(A) \xleftarrow{\mathrm{cons}} A \times \mathrm{List}(A)$$

Arithmetic universes (AUs)

intrinsic infinities
e.g. N = List(1)

Aims

- Finitary formalism for geometric theories

- Dependent type theory of (generalized) spaces

- Use methods of classifying toposes in base-independent way

- Computer support for that

- Foundationally very robust - topos-valid, predicative

- Logic intemalizable in itself
  (cf. Joyal applying AUs to Goedel's theorem)

Classifying AUs

Universal algebra => AUs can be presented by
- generators (objects and morphisms)
- and relations

theory of AUs is cartesian
(essentially algebraic)

(G, R) can be used as a logical theory

AU<G|R> has property like that of classifying toposes

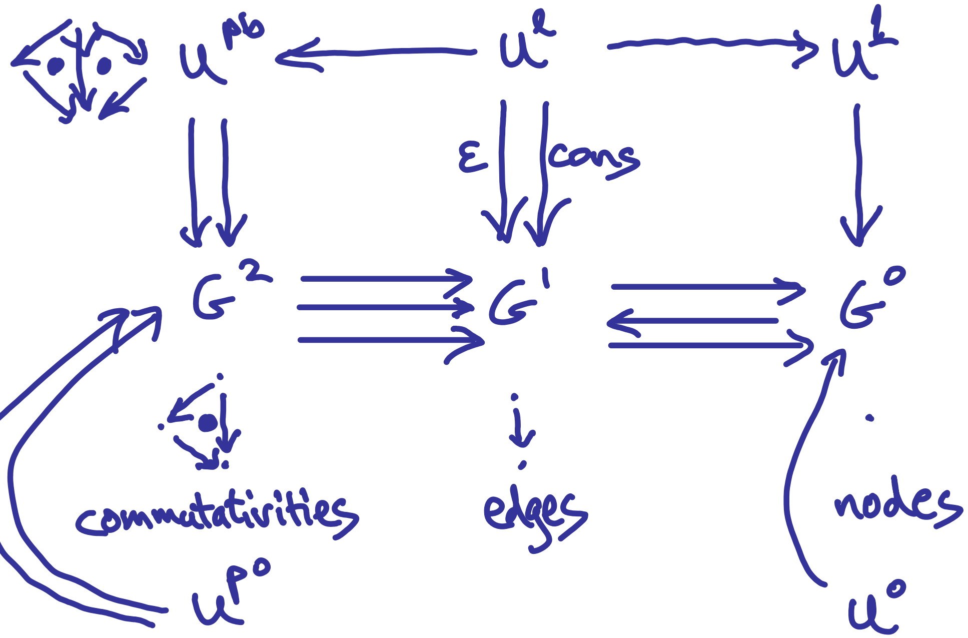
Treat AU<G|R> as "space of models of (G,R)"
- But no dependence on a base topos!

Issues: How to present theories?

"Arithmetic" instead of geometric

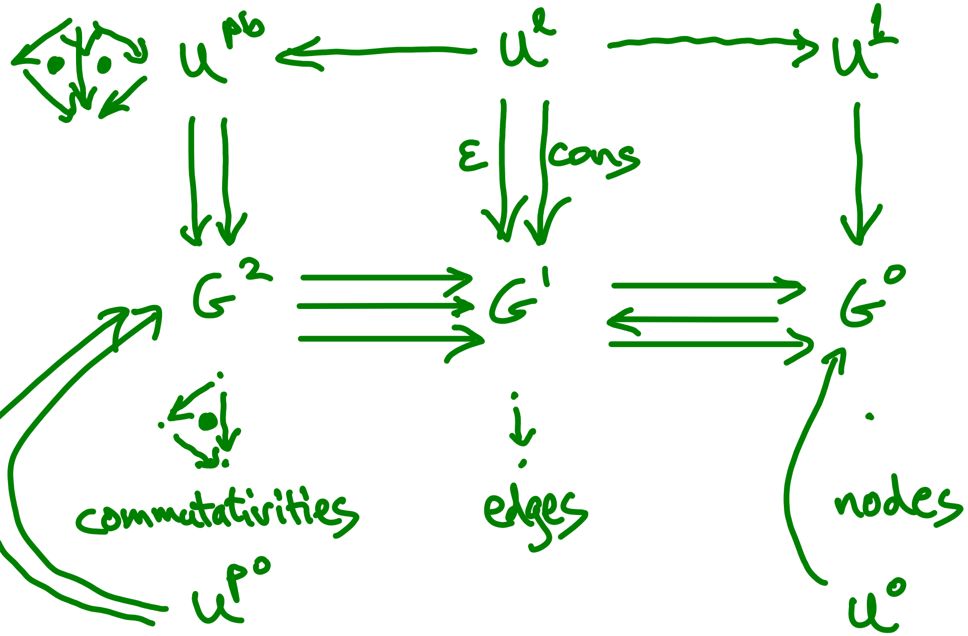
Not pure logic - needs ability to construct new sorts, e.g. N, Q

Use sketches - hybrid of logic and category theory
- sorts, unary functions, commutativities
- universals: ability to declare sorts as finite limits, finite colimits or list objects



e.g. binary operations (M, m)

$$\begin{tikzcd} M^2 \arrow[loop above, "id"] \arrow[r, shift left, "p_i"] \arrow[r, shift right, "m"'] & M \arrow[loop above, "id"] \arrow[r, "!"] & 1 \arrow[loop above, "id"] \end{tikzcd}$$



$G^0 = \{1, M, M^2\}$

$G^1 = \{id, id, id, p_1, p_2, m, !\}$

$G^2 = \left\{ \begin{tikzcd} M^2 \arrow[d, "p_1"'] \arrow[dr, "!"] & \\ M \arrow[r, "!"'] & 1 \end{tikzcd} , \begin{tikzcd} M^2 \arrow[dr, "p_2"] \arrow[d, "!"'] & \\ 1 & M \arrow[l, "!"] \end{tikzcd} \right\}$

$U^1 = \{1\}$

$U^{pb} = \left\{ \begin{tikzcd} & M^2 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, "!"] & \\ M \arrow[dr, "!"'] & & M \arrow[dl, "!"] \\ & 1 & \end{tikzcd} \right\}$

Issues: strictness

Strict model - interprets pullbacks etc. as the canonical ones
- needed for universal algebra of AUs

But non-strict models are also needed for semantics

Contexts are sketches built in a constrained way
- better behaved than general sketches
- every non-strict model has a canonical strict isomorph

Con is 2-category of contexts
- made by finitary means

A base-independent category of (some) generalized point-free spaces

The assignment T |-> AU<T>
is full and faithful 2-functor
- from contexts
- to AUs and strict AU-functors (reversed)

"Sketches for arithmetic universes"

Models in toposes

- but the same works for models in AUs

Suppose T a context (object in Con),

E an elementary topos with nno

Then have category E-Mod-T of strict T-models in E

If H: T1 -> T2 a context map (1-cell in Con), then get

map H as model transformer

E-Mod-H: E-Mod-T1 -> E-Mod-T2,   M |-> MH

$$\mathcal{E} \xleftarrow{\;M\;} \mathrm{Au}\langle \mathbb{T}_1 \rangle \xleftarrow{\;\mathrm{Au}\langle H \rangle\;} \mathrm{Au}\langle \mathbb{T}_2 \rangle$$

2-cells give natural transformations

E-Mod is strict 2-functor Con -> Cat

Models in different toposes

If f: E1 -> E2 a geometric morphism,
then inverse image part f*: E2 -> E1 is a non-strict AU-functor

We get

f-Mod-T: E2-Mod-T -> E1-Mod-T,   M |-> f*M

Apply f* (giving non-strict model), and then take canonical strict isomorph

f |-> f-Mod-T is strictly functorial!

toposes with nno,
geometric morphisms

Mod-T is a strictly indexed category over Top

Bimodule identity

In general:

$(f^*M)H$ isomorphic to $f^*(MH)$

However, for certain well-behaved H (extension maps) have

$(f^*M)H = f^*(MH)$

Extension maps also have strict pullbacks along all 1-cells in Con

Bundles

$$\begin{array}{c} \mathbb{T}_1 \\ \big\downarrow U \\ \mathbb{T}_0 \end{array}$$

U an extension map (in Con)
As map, U transforms models:
T_1 models N
  |-> T_0 model NU

Bundle view says U transforms T_0 models to spaces, the fibres:

M |-> "the space of models N of T_1 such that NU = M"

Suppose M is a model in an elementary topos (with nno) S.
Then fibre exists as a generalized space in Grothendieck's sense
- get geometric theory T_1/M (of T_1 models N with NU = M)
- it has classifying topos

$$\begin{array}{c} \mathcal{S}[\mathbb{T}_1/M] \\ \big\downarrow \\ \mathcal{S} \end{array}$$

"Arithmetic universes and classifying toposes":

all fibred over 2-category of pairs (S, M)

Change of S

Get pseudopullback -

$$
\begin{tikzcd}
\mathcal{S}'[\pi_1/f^*M] \arrow[r] \arrow[d] & \mathcal{S}[\pi_1/M] \arrow[d, "p"] \\
\mathcal{S}' \arrow[r, "f"'] & \mathcal{S}
\end{tikzcd}
$$

bounded

not necessarily bounded

Example: local homeomorphisms

Theories of sets and of pointed sets can be expressed with a context extension map

$$\begin{array}{l} [\mathbb{O}, pt] \\ \big\downarrow u \\ [\mathbb{O}] \end{array}$$

one sort,
one constant

one sort

$$\begin{array}{l} \mathcal{S}[\mathbb{O}, pt / X] \\ \big\downarrow p \\ \mathcal{S} \end{array}$$

Model of [O] in S is object X of S
S[O,pt / X] is discrete space for X over S

p is a local homeomorphism

Every local homeomorphism between elementary toposes with nno can be got this way - not dependent on choosing some base topos

Conclusions

Con is proposed as a category of a good fragment of Grothendieck's generalized spaces
- but in a base-independent way
- consists of what can be done in a minimal foundational setting
- of AUs
- constructive, predicative
- includes real line

Current work (with Sina Hazratpour)

- use calculations in Con to prove fibrations and opfibrations in Top.

References for AUs

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