The frame of Scott continuous nuclei on a preframe

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Opportunity to tell a single story based on several papers

- With a few new unwritten results.
- With some open problems for which I've ran out of tools and ideas.

  Perhaps some you will be able to tackle them.

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Chronology of the selected papers and research notes for this story:

1. Properly injective spaces and function spaces.
2. On the compact-regular coreflection of a stably compact locale.
3. Spatiality of the patch frame.
4. The patch frame of the Lawson dual of a stably continuous frame.
5. The regular-locally-compact coreflection of a stably locally compact locale.
6. (Function-space compactifications of function spaces.)
7. Joins in the frame of nuclei.
8. Compactly generated Hausdorff locales.

The story here won't be strictly chronological or complete, though.

(Non) constructivity in this story

1. Everything I say about topological spaces is non-constructive.

   It uses at least excluded middle, and often choice too.

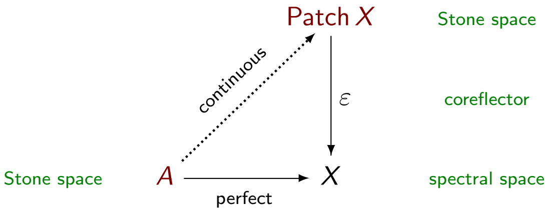
2. Everything I say about locales is constructive.

   It works in any topos.

- I won't discuss why it is fun and advantegeous to be constructive.
- That would be the subject of another story.

  Perhaps you will hear about this from me in another opportunity.

Patch a spectral space to get a Stone space

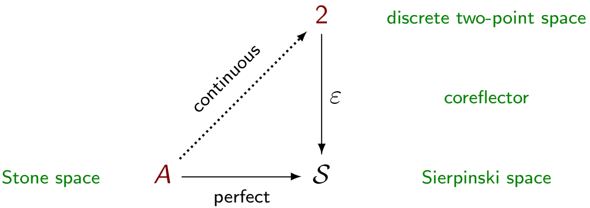


perfect = continuous & inverse images of compact opens are compact.

- Continuous maps of Stone spaces are automatically perfect.
- Hence they form a coreflective subcategory of the category of perfect maps of spectral spaces.

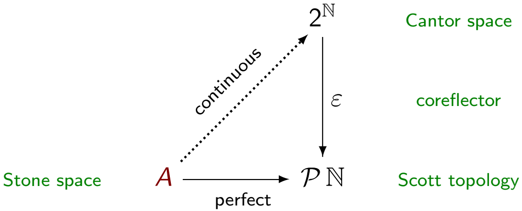
Example: patch the Sierpinsi space $\mathcal{S}$ to get the the two-point discrete space 2

$\mathcal{S}$ = one limit point $\bot$, one isolated point $\top$.
$2 = 1 + 1$ = two isolated points $\bot$ and $\top$.



- Continuous maps into 2 classify clopens.
- Continuous maps into Sierpinski classify opens.
- Perfect maps into Sierpinski classify clopens.
- Discrete topology = Lawson topology of the order $\bot \leq \top$.
  Sierpinski topology = Scott topology of that order.

Related example:



- Cantor topology = product topology = Lawson topology.

Stably compact spaces

Sober, compact, locally compact, with compact saturated sets closed under finite intersections. Equivalently:

1. The retracts of spectral spaces.
2. The algebras of the prime-filter monad on T_0 topological spaces.

   (The *free* algebras are the *spectral spaces*.)
3. The injective T_0 spaces w.r.t. flat embeddings.

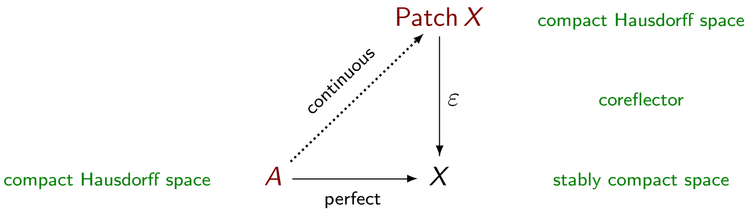
   An embedding is flat iff all finite T_0 spaces injective along it.

   (Isbell. "Flat = prosupersplit". 1988)

Non-spectral examples:

- Unit interval $[0, 1]$ with the topology of lower semicontinuity.
- Closed interval of $[0, 1]$ with the Scott topology of reverse inclusion.
- More generally continuous Scott domains with Scott topology.
- Even more generally Jung's FS domains.
- Probabilistic powerdomains of FS domains.

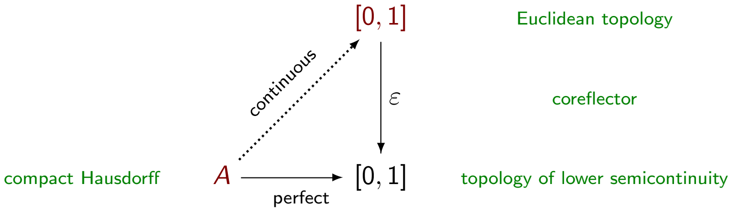
Patch a stably compact space to get a compact Hausdorff space



perfect = continuous & inverse images of compact saturated sets are compact.

- Continuous maps of compact Hausdorff spaces are automatically perfect.
- Hence they form a coreflective subcategory of the category of perfect maps of stably compact spaces.

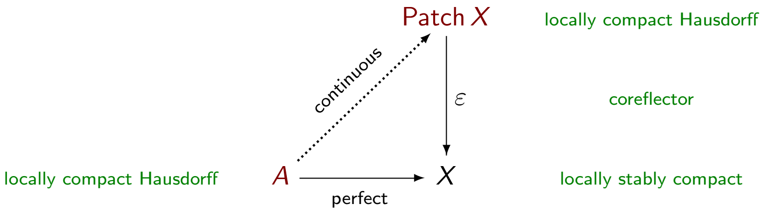
Example: patch the topology of lower semicontinuity to get the Euclidean topology



- Euclidean topology = Lawson topology of the natural order.
- Topology of l.s.c. = Scott topology of the natural order.
- perfect = lower and upper semicontinuous = continuous.

Patch a locally stably compact space to get a locally compact Hausdorff space

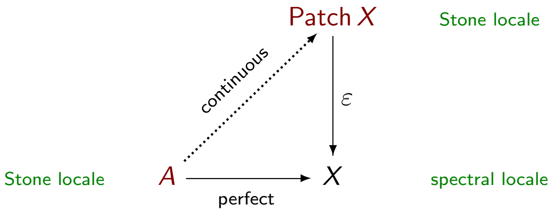
Drop compactness in the definition of stable compactness.



How to patch a locale

1. Nucleus = closure operator that preserves finite meets
   $\cong$ quotient frame $\cong$ sublocale = regular mono.
2. Scott continuous nucleus = preserves directed joins.
3. $\mathcal{O}\,\mathsf{Patch}\,X$ = Scott continuous nuclei on $\mathcal{O}\,X$.
4. $\varepsilon^*(U)$ = closed nucleus induced by the open U, namely $U \vee (-)$.

Theorem. Continuous maps of Stone locales form a coreflective subcategory of perfect maps of spectral locales:



Corollary. The patch topology of a spectral space is isomorphic to the frame of Scott continuous nuclei on the given topology.

Independently, Panagis Karazeris proved directly the corollary, without considering the theorem.

The frame of Scott continuous nuclei on a preframe

- Preframe. Poset with finite meets (including a top element) and directed joins with $u \wedge \bigvee_i v_i = \bigvee_i u \wedge v_i$ for all directed $(v_i)_i$.
- Frame. Also has finite joins which distribute over finite meets.

Theorem.

1. The Scott continuous nuclei on a preframe L form a frame.
2. And a subframe of the frame of all nuclei if L is a frame.

Proof

- Finite meets and directed joins of are computed pointwise.
- Finite joins are computed as follows.

  (i) Compute the finite compositions of the nuclei.

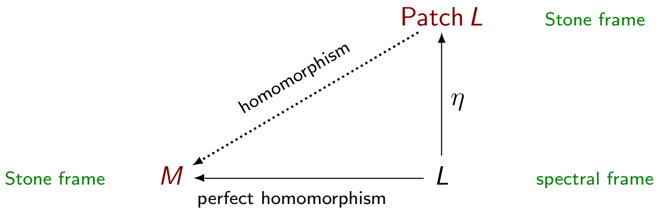
  (ii) The resulting functions are not nuclei in general.

  (iii) But they form a directed set whose pointwise join is a nucleus.

Patch in the language of frames

1. Patch L = Scott continuous nuclei on the frame L.
2. $\eta(U)$ = closed nucleus induced by the open U.

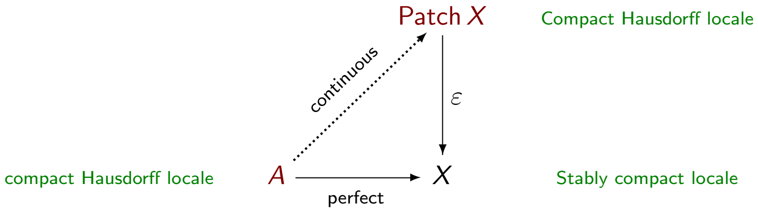
Theorem. This construction gives the universal solution to the problem of adding boolean complements to the compact elements of a spectral frame, to get a Stone frame:



Because homomorphisms of Stone frames are perfect.

Patching a stably compact locale

- $\mathcal{O}\,\mathrm{Patch}\,X$ = Scott continuous nuclei on $\mathcal{O}\,X$.
- $\varepsilon^*(U)$ = closed nucleus induced by the open U.

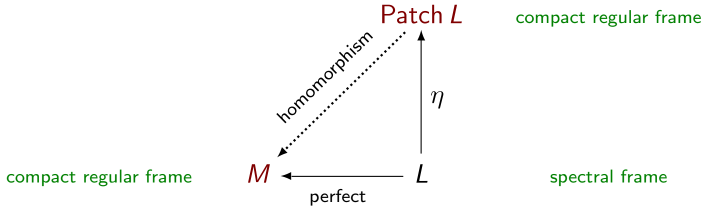


- (Strongly) Hausdorff = the diagonal map $X \to X \times X$ is closed.
- locally compact = every open U is the join of the opens $V \ll U$.
- The way-below relation $V \ll U$ means that every open cover of U has a finite subcover of V.
- Stably compact: locally compact, $1 \ll 1$ and $U \ll V$ and $U \ll W$ together imply $U \ll V \wedge W$.

In the language of frames

1. Perfect homomorphism = its right adjoint is Scott continuous.
2. For locally compact locales $\iff$ the homomorphism preserves $\ll$.
3. Compact Hausdorff = compact regular.
4. Regular = every open U is the join of the opens $V \leqslant U$.
5. The well-inside relation $V \leqslant U$ means $U \vee \neg V = 1$.
   (We can interpolate a closed sublocale between V and U.)
6. In a compact regular locale, $\ll$ coincides with $\leqslant$.
7. Frame homomorphisms (of any two frames) preserve $\leqslant$.

Theorem. The frame of Scott continuous nuclei on a stably compact frame gives the universal solution to the problem of transforming $\ll$ into $\leqslant$, to get a compact regular frame:



Patching a locally stably compact frame

- Locally stably compact. Drop $1 \ll 1$ (compactness) from the definition of stable compactness.

Theorem. The frame of Scott continuous nuclei on a locally stably compact frame gives the universal solution to the problem of transforming

$$U \ll V$$

into

$$U \leqslant V \ll 1$$

to get a locally compact Hausdorff frame.

The Lawson dual $L^\wedge$ of a preframe L

1. $L^\wedge$ = Scott open filters on L.

   Again a preframe.

2. The *natural preframe homomorphism* $\eta : L \to L^{\wedge\wedge}$ maps $U \in L$ to the Scott open filter of filters $\{\phi \in L^\wedge \mid U \in \phi\}$.

Theorem. Let L be a stably compact frame.

1. $L^\wedge$ is again a stably compact frame.
2. The natural map $L \to L^{\wedge\wedge}$ is an isomorphism.
3. L and $L^\wedge$ have the same patch.

Compactly generated Hausdorff spaces

1. Introduced by Hurewicz around 1948 (unpublished) because he wanted a cartesian closed category of spaces.
   - Before the notion of cartesian closed category was formulated.
   - But often named after John Kelley because of his 1955 book.
   - Popularized by Steenrod 1967 as a convenient category of spaces.
2. Homotopies $[0,1] \times X \to Y$ correspond to paths $[0,1] \to Y^X$ in function spaces.
3. Continuous $A \times X \to Y$ correspond to continuous $A \to Y^X$.

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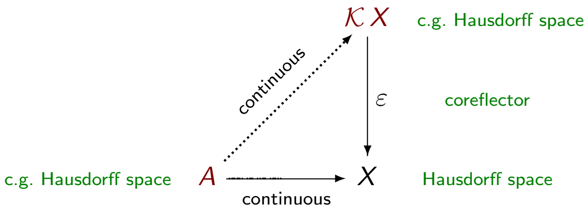
3. Continuous $A \times X \to Y$ correspond to continuous $A \to Y^X$.
4. Trouble: there is in general no way to topologize the set of continuous functions Y^X to get this correspondence, not even when $A = [0,1]$.

   (Day and Kelly showed in 1970 that it is necessary and sufficient that $\mathcal{O}X$ be a continuous lattice in the sense of Dana Scott.)
5. But this works for the category of compactly generated spaces.
6. In modern language, compactly generated space = colimit of compact Hausdorff spaces.

Compactly generated Hausdorff spaces



- Use to prove their cartesian closedness.
- Can we do the same thing for locales?

  Some of it for the moment.

Compactly generated proto-Hausdorff locales

- Proto-Hausdorff = every compact sublocale is closed and Hausdorff.
- Let $\mathcal{K}X$ be the colimit of the diagram of compact sublocales of X with inclusions, and let $\varepsilon : \mathcal{K}X \to X$ be the natural map.
- We define X to be compactly generated iff ε is an isomorphism.

Theorem. For X proto-Hausdorff:

1. $\mathcal{O}\mathcal{K}X \cong$ frame of Scott continuous nuclei on the preframe $(\mathcal{O}X)^\wedge$ $\cong (\mathcal{O}X)^{\wedge\wedge}$
2. X is compactly generated iff the natural map $\mathcal{O}X \to (\mathcal{O}X)^{\wedge\wedge}$ is an isomorphism.
3. X is compactly generated iff $\mathcal{O}X \cong (\text{compact sublocales of } X \text{ ordered by reverse inclusion})^\wedge$
4. The map $\varepsilon : \mathcal{K}X \to X$ is a coreflector into compactly generated proto-Hausdorff locales.

What happens in the Hausdorff case?

Sufficient conditions for the natural map $\varepsilon : \mathcal{K}X \to X$ to be a coreflector:

1. It is a monomorphism for all X.
2. $\mathcal{K}$ preserves the Hausdorff property. (Implied by the above.)

These are open questions.

Difficulties in trying to get a cartesian closed category of compactly generated locales

Incomplete recipe to construct Y^X for X compactly generated and Y (proto-)Hausdorff:

1. Decompose X into its building compact Hausdorff blocks X_i.
2. The exponentials Y^{X_i} exist (Hyland, "Function spaces in the category of locales".)
3. Take the limit, because $Y^{(-)}$ ought take colimits to limits.
4. Take Y^X to be the compactly generated reflection of this limit.
5. Use the universal properties at our disposal (those of Y^{X_i} and of the coreflection) to prove the desired universal property of Y^X.

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This works for spaces.

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This works for spaces.

- However, for the localic exponential Y^{X_i} to be Hausdorff if Y is, we need X to be "open". (Johnstone, "Open locales and exponentiation".)
- We can try to get rid of the (proto-)Hausdorff condition.
- But then there are difficulties in finding a canonical small category of "building blocks" to construct a compactly generated reflection.