The variety of nuclear implicative semilattices is locally finite

Guram Bezhanishvili, Nick Bezhanishvili, David Gabelaia, Silvio Ghilardi, Mamuka Jibladze

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for any $a, b, c \in A$.

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for all $a, b \in A$.

A less concise but probably more understandable equivalent formulation:

- $a \leqslant \mathbf{j}\, a$
- $\mathbf{j}\,\mathbf{j}\, a = \mathbf{j}\, a$
- $\mathbf{j}(a \wedge b) = \mathbf{j}\, a \wedge \mathbf{j}\, b$

Terminology – $\mathbf{j}$ is a *nucleus*.

First appearance?

(F. W. Lawvere, "Toposes, Algebraic Geometry and Logic", Introduction. Dalhousie University, Halifax 1971, Springer LNM 274)

such that when the diagram is pulled back to $\underline{E}/S$ it has the property in the sense of the topos $\underline{E}/S$. The other notion is with respect to a given $\Omega \xrightarrow{j} \Omega$ which may be thought of as a modal operator to be read "it is j-locally the case that .." and which satisfies the axioms below which in particular mean that j is equivalent to a Grothendieck topology on $\underline{C}$ in the case of a topos of the form $S^{\underline{C}^{op}}$. At the Rome and Overwolfach meetings I had pointed out that the usual notion of a Grothendieck topology is equivalent to a single such morphism j ; Tierney showed that the appropriate axioms on j are simply that $jj = j$ and j preserves finite conjunctions.* A subobject $X' \rightarrowtail X$ with characteristic function $X \xrightarrow{\phi} \Omega$ is said to be j-dense if ϕ is j-locally true

Main contributors

Lawvere and Tierney used nuclei to interpret (Cohen) forcing in their topos-theoretic proof of independence of the Continuum Hypothesis.

Roughly, the forcing relation $p \Vdash \varphi$ between a poset of forcing conditions and formulæ of certain theory corresponds in their context to $p \in \mathbf{j}(\mathrm{Val}(\varphi))$ (where $\mathrm{Val}()$ is the valuation in a given model of certain theory).

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Isbell, Simmons, Banaschewski, Johnstone, Pultr, Picado, Escardo, ...

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Thus cHa semantics (in particular, topological semantics) for intuitionistic logic can be reformulated using Kripke models with extra structure, in form of a nucleus.

Dragalin frames

The latter had several alternative descriptions in the literature - The idea of coverage (Johnstone): $U \triangleright x$ (or $x \triangleleft U$), a relation between elements and up-sets of P, re-axiomatizing "$x \in \mathbf{j}\,U$" ("**j** makes elements of U cover p").

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(He only had it for topological semantics; recently generalized by Guram Bezhanishvili and Wesley Holliday to any complete Heyting algebras.)

Semantics for Propositional Lax Logic

(Journal of Logic and Computation **21** (2011), pp. 1035–1063)

Cover semantics for quantified lax logic

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Abstract

Lax modalities occur in intuitionistic logics concerned with hardware verification, the computational lambda calculus and access control in secure systems. They also encapsulate the logic of Lawvere–Tierney–Grothendieck topologies on topoi. This article provides a complete semantics for quantified lax logic by combining the Beth–Kripke–Joyal cover semantics for first-order intuitionistic logic with the classical relational semantics for a 'diamond' modality. The main technique used is the lifting of a multiplicative closure operator (nucleus) from a Heyting algebra to its MacNeille completion, and the representation of an arbitrary locale as the lattice of 'propositions' of a suitable cover system. In addition, the theory is worked out for certain constructive versions of the classical logics K and S4. An alternative completeness proof is given for (non-modal) first-order intuitionistic logic itself with respect to the cover semantics, using a simple and explicit Henkin-style construction of a characteristic model whose points are principal theories rather than prime-saturated ones. The article provides further evidence that there is more to intuitionistic modal logic than the generalization of properties of boxes and diamonds from Boolean modal logic.

Keywords: Modality, quantified lax logic, intuitionistic logic, locale, nucleus, MacNeille completion, cover system, refinement, local membership, Beth-Kripke-Joyal semantics, fallible world, Constructive S4.

1 The curious concept of lax modality

Propositional lax logic (PLL) was defined by Fairtlough and Mendler [13, 14] as an intuitionistic propositional logic with a modality $\bigcirc$ having the axioms

$$\varphi \to \bigcirc\varphi$$
$$\bigcirc\bigcirc\varphi \to \bigcirc\varphi$$
$$\bigcirc\varphi \wedge \bigcirc\psi \to \bigcirc(\varphi \wedge \psi).$$

The motivation was hardware specification and verification, with a modal formula $\bigcirc\varphi$ being read 'there is some constraint under which φ is true'. Typical constraints are timing delays on input signals to digital circuits, with the first axiom corresponding to a single wire with no delay; the second corresponding to the sequential composition of circuits, with constraint given by adding their delays; and the third axiom corresponding to parallel composition, with the maximum of the two delays as constraint. The term 'lax' was chosen 'to indicate the looseness associated with the notion of correctness up to constraints' [14, p. 3].

In fact there have been several independently motivated investigations that have produced an intuitionistic logic with a *lax modality*, i.e. one having the above axioms.[1] The first would appear to be a Gentzen-style calculus studied by Curry [9] for proof-theoretic purposes. Its modality, denoted $\diamond$, was intended to express possibility.

[1]These studies are also reviewed in Section 7.6 of the author's historical survey [23].

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Examples

Open nuclei

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"*Quasi-closed*" nuclei

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On a Boolean algebra, every nucleus $\mathbf{j}$ has form

$$\mathbf{j}\,x = a \vee x$$

(fixed points $\uparrow(a)$).

Köhler duality

Our proof of local finiteness of the variety of nuclear implicative semilattices is based on the duality for *finite* implicative semilattices developed in

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Every finite implicative semilattice is isomorphic to one of the form $\mathrm{Up}(X)$ for a finite partially ordered set X.

Köhler duality

Moreover homomorphisms $h : \mathrm{Up}(X') \to \mathrm{Up}(X)$ are determined by certain partial maps

$$X \supseteq Y \xrightarrow{f} X';$$

namely, Y can be arbitrary subset of X while $f : Y \to X'$ is a *strict p-morphism*.

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Such a p-morphism is called strict if moreover $y_0 < y_1$ implies $f(y_0) < f(y_1)$ for all $y_0, y_1 \in Y$.

Köhler duality

A partial strict p-morphism

$$X \supseteq Y \xrightarrow{f} X'$$

gives rise to an implicative semilattice homomorphism $h_f : \mathrm{Up}(X') \to \mathrm{Up}(X)$.

$$\mathrm{Up}(X) \xrightarrow{Y \cap -} \mathrm{Up}(Y) \xleftarrow{f^{-1}} \mathrm{Up}(X')$$

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\mathrm{Up}(X) \arrow[r, two heads, "Y \cap -"] & \mathrm{Up}(Y) \arrow[l, bend right] & \mathrm{Up}(X') \arrow[l, "f^{-1}"']
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and every implicative semilattice homomorphism
$h : \mathrm{Up}(X') \to \mathrm{Up}(X)$ has this form for a unique partial strict p-morphism f.

Köhler duality and nuclei

We first extend the Köhler duality to *nuclear* finite implicative semilattices $(\mathrm{Up}(X), \mathbf{j})$ where $\mathbf{j}$ is a nucleus on $\mathrm{Up}(X)$.

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Now every subset $S \subseteq X$ of a poset X gives rise to a nucleus $\mathbf{j}_S$ on $\mathrm{Up}(X)$,

$$\mathbf{j}_S(U) = X \smallsetminus {\downarrow}(S \smallsetminus U),$$

and for finite posets X, every nucleus $\mathbf{j} : \mathrm{Up}(X) \to \mathrm{Up}(X)$ is equal to $\mathbf{j}_S$ for a unique $S \subseteq X$.

Köhler duality and nuclei

Then, to complete the extension of the Köhler duality to nuclear implicative semilattices, we need to answer this question:

Given finite posets X, X' and subsets $S \subseteq X$, $S' \subseteq X'$, for which partial strict p-morphisms

$$X \supseteq Y \xrightarrow{f} X'$$

is the corresponding implicative semilattice homomorphism $h_f : \mathrm{Up}(X') \to \mathrm{Up}(X)$ actually a homomorphism of nuclear implicative semilattices $(\mathrm{Up}(X'), \mathbf{j}_{S'}) \to (\mathrm{Up}(X), \mathbf{j}_S)$?

Dual description of subalgebras

Having obtained description of such homomorphisms, we in particular obtain a dual description of subalgebras of a nuclear implicative semilattice $(\mathrm{Up}(X), \mathbf{j}_S)$.

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Nuclear implicative subsemilattices of a finite nuclear implicative semilattice $(\mathrm{Up}(X), \mathbf{j}_S)$ *are in one-to-one correspondence with partial equivalence relations* $\sim$ *on* X *with the following properties:*

- $\sim$ *is p-morphic:* $\forall\ y_1 \sim y_2 \leqslant y_2'\ \exists\ y_1 \leqslant y_1' \sim y_2'$;

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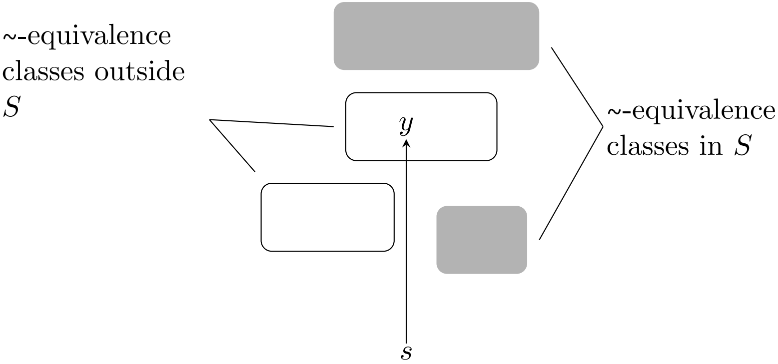
- S *is* $\sim$*-saturated: every* $\sim$*-equivalence class is either disjoint from* S *or contained in* S*;*
- *for all* $s \in S$ *and all* $y \geqslant s$*, if* y *belongs to a* $\sim$*-equivalence class, then there is an* $s' \in S$ *belonging to a (possibly different)* $\sim$*-equivalence class, such that* $s \leqslant s'$ *and* $s' \leqslant y'$ *with* $y \sim y'$.

Dual description of subalgebras



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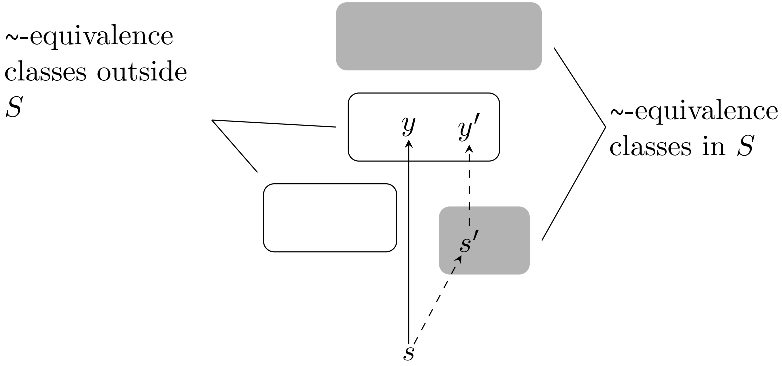

General method of Ghilardi

Armed with this dual description of subalgebras, we can now use the powerful general method of description of universal models given in

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In that paper, for any variety having the variety of implicative semilattices as a reduct, a fairly explicit construction of universal models is given *provided* one knows the dual description of the situation when a finite algebra A is generated by its given elements a_1, ..., a_n.

Generators

We thus also need to answer the following question:

Given a finite poset X, a subset $S \subseteq X$, and up-sets $U_1, ..., U_n \in \mathrm{Up}(X)$, when does it happen that these up-sets generate $\mathrm{Up}(X)$ as a nuclear implicative semilattice? That is, $(\mathrm{Up}(X), \mathbf{j}_S)$ does not possess any proper nuclear implicative subsemilattices $A \subseteq \mathrm{Up}(X)$ containing all U_1, ..., U_n?

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To each element $x \in X$ one assigns a set $\mathrm{c}(x)$ of “colors”, to determine uniquely to which of the $U_1, ..., U_n$ does x belong.

For example, if x belongs to U_1 and U_2 and does not belong to any other U_i, one puts $\mathrm{c}(x) = \{1, 2\}$.

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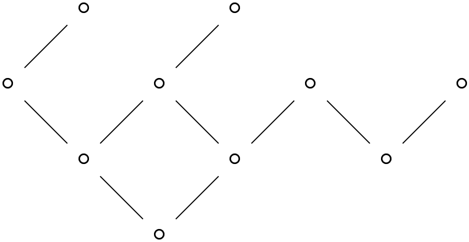
In terms of these colors one can give a dual description of the situation when $U_1, ..., U_n$ generate $\mathrm{Up}(X)$ as a nuclear implicative semilattice. Ghilardi calls the corresponding dual "colored" models *irreducible*.

Coloring and S



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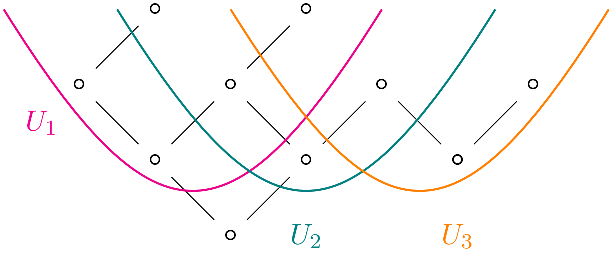

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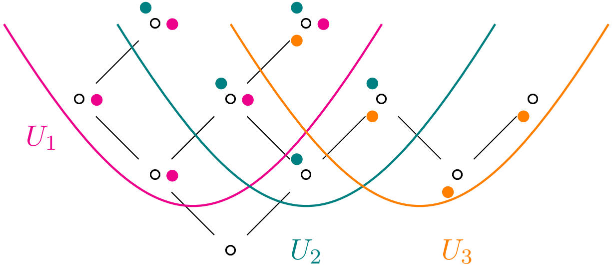

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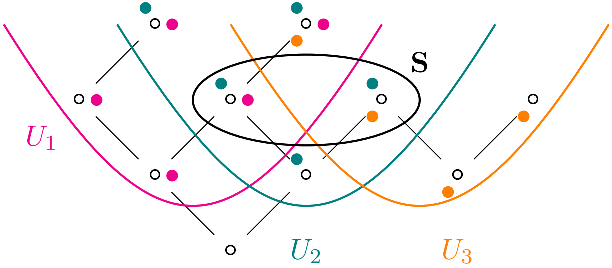

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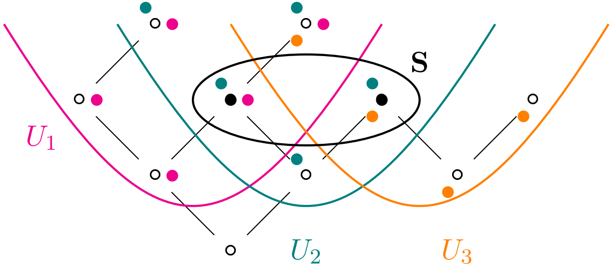

Colorings and irreducibility

To characterize irreducibility, let us also extend the colorings from elements to subsets of X via

$$\mathrm{c}(X') = \bigcap \left\{ \mathrm{c}(x) \mid x \in X' \right\}$$

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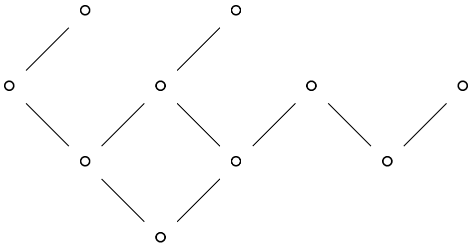
for $X' \subseteq X$.

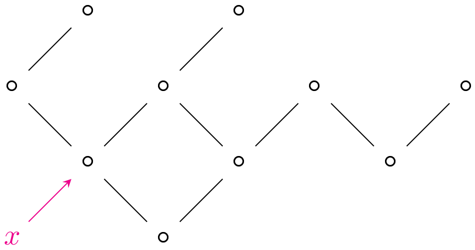
Moreover, for $x \in X$ let

$$\triangledown x := \min \left(\uparrow(x) \smallsetminus \{x\} \right).$$

We then have

Theorem. *The up-sets* $U_1, \ldots, U_n$ *of* $\mathrm{Up}(X)$ *generate* $(\mathrm{Up}(X), \mathbf{j}_S)$ *as a nuclear implicative semilattice if and only if in the corresponding colored model,* $\mathrm{c}(x) = \mathrm{c}(\triangledown x)$ *implies that* $x \in S$ *and moreover* $\triangledown x \notin S$.

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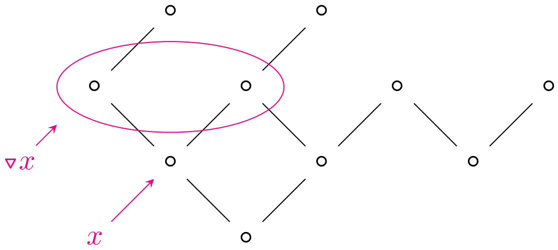
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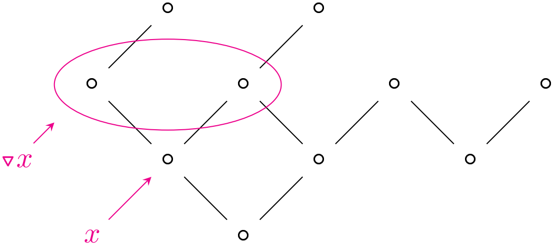

x retains all possible colors $\Rightarrow$ x is black, and $\triangledown x$ is not all black.

Universal model

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The universal n-model $L = (X, S, U_1, ..., U_n)$ is an (a priori infinite) poset with n upsets U_1, ..., U_n is characterized by the property that for any *finite* $(X', S', U_1', ..., U_n')$ there is a *unique* embedding $X' \rightarrowtail X$ which identifies X' with an up-set of X such that under this identification, $S' = X' \cap S$, $U_i' = X' \cap U_i$, $i = 1, ..., n$.

Universal model - stepwise construction

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Moreover if $\alpha \nsubseteq S_k$ then we also add $s_{\alpha,\mathrm{c}(\alpha)} \in S_{k+1}$.

Universal model - stepwise construction

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Moreover if $\alpha \nsubseteq S_k$ then we also add $s_{\alpha,\mathrm{c}(\alpha)} \in S_{k+1}$.

The partial order is extended from X_k to X_{k+1} by the equalities $\triangledown r_{\alpha,\sigma} = \triangledown s_{\alpha,\sigma} = \alpha$ and the valuation by $\mathrm{c}(r_{\alpha,\sigma}) = \mathrm{c}(s_{\alpha,\sigma}) = \sigma$ (including $s_{\alpha,\mathrm{c}(\alpha)}$ whenever it exists).

Finiteness of the universal model

Let then (X_∞, S_∞) be the union of the expanding sets X_k and S_k. Then L is the model $(X_\infty, S_\infty, \mathrm{c})$, with c extended all along the X_k.

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Proof of finiteness of L is based on the above characterization of irreducibility. This characterization gives two things: on the one hand, when $x \notin S$, we must have $\mathrm{c}(x) \subsetneq \mathrm{c}(\triangledown x)$, so that if there were no S, we would run out of colors after depth n (= number of generators). This can be used to show that $X \smallsetminus S$ is actually finite.

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On the other hand, although for $s \in S$ one is allowed to have $\mathrm{c}(s) = \mathrm{c}(\triangledown s)$, still each such s is required to possess some element $r \in (\triangledown s) \smallsetminus S$.

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Combining these facts, one manages to show that along any descending chain $s_1 > s_2 > s_3 > \ldots$ in S one eventually runs out of required elements $r_k \in (\triangledown s_k) \smallsetminus S$.

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This then can be used to prove finiteness of L.

Finite model property

There is a more or less standard argument to show that in presence of the finite model property, finiteness of the universal n-model for every n implies local finiteness of the variety.
Finite model property for the variety of nuclear implicative semilattices is relatively easy to show.

What remains to be done

It would be much nicer of course to have a purely algebraic proof of local finiteness. There is a very simple such proof for implicative semilattices, based on induction on the number of generators and on properties of subdirectly irreducible algebras.

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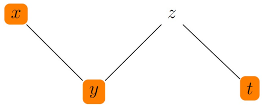
From local finiteness one must be able to obtain normal forms for nuclear implicative semilattices.

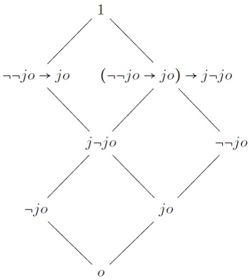
What remains to be done

How fast does it grow?

For $n = 1$, the cyclic algebra is easy to describe; the dual looks like



(with $S = \{x, y, t\}$), and the algebra itself is



(with o the generator and $\neg = (\_ \to o)$).

What remains to be done

However already the 2-generated algebras may be huge. Experimentally, the dual can have up to six elements on the highest (zeroth) level, up to 68 elements on the next (first) level, and at least billions of elements on the second level. We only have a rough upper bound for its depth.

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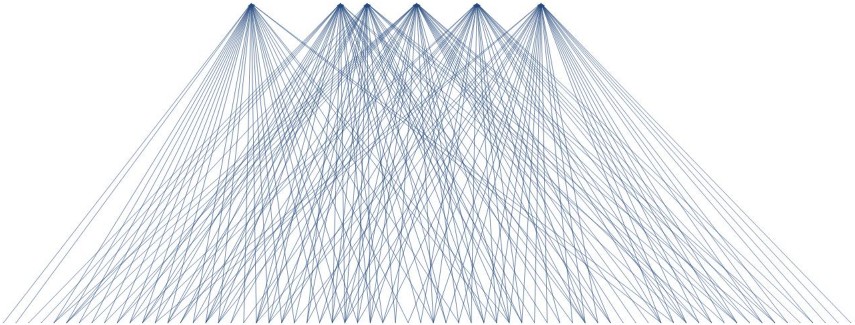

Thank you for patience!