A new characterization of the class $\mathrm{HSP_U}(\mathcal{K})$

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Finite Embeddability Property

Definition

A class of algebras $\mathcal{K}$ has the finite embeddability property (FEP) if every finite partial subalgebra of any algebra from $\mathcal{K}$ can be embedded into a finite member of $\mathcal{K}$.

- useful when dealing with the word problem
- applications in logic

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A different approach to FEP

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An algebra $\mathcal{A}$ satisfies the generalized finite embeddability property (GFEP) for a class $\mathcal{K}$ of algebras of the same type if every finite partial subalgebra of $\mathcal{A}$ can be embedded into an algebra from $\mathcal{K}$.

Theorem

An algebra $\mathcal{A}$ satisfies GFEP for $\mathcal{K}$ if and only if $\mathcal{A} \in \mathrm{ISP_U}(\mathcal{K})$.

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Finite Coverability Property

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Let $\mathcal{A} = (A, F)$ be an algebra and $\mathcal{K}$ be a class of algebras of the type F. We say $\mathcal{A}$ satisfies *the finite coverability property for the class* $\mathcal{K}$ if for every finite set of terms $T \subseteq T_F(A)$ there exist an algebra $\mathcal{B} \in \mathcal{K}$, a mapping $f : B \to A$ and a set $Y \subseteq B$ such that

- $f|_Y : Y \to \mathrm{Var}\, T$ is a bijection,
- if $t(a_1, \dots, a_n) \in T$ and $y_1, \dots, y_n \in Y$ are such that $fy_i = a_i$ then

$$ft^{\mathcal{B}}(y_1, \dots, y_n) = t^{\mathcal{A}}(a_1, \dots, a_n).$$

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Sketch of proof: There exists an ultrafilter $\mathcal{U}$ on the set $\mathcal{P}_{\text{fin}} T_F(A)$ such that it contains all sets

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For every $T \in \mathcal{P}_{\text{fin}} T_F(A)$ there exist an algebra $\mathcal{B}_T \in \mathcal{K}$, a mapping $f_T \colon B_T \to A$ and a set $Y_T \subseteq B_T$ satisfying the conditions of FCP.
Let $a \in \mathcal{A}$ then $y \in \prod_T Y_T$ is called *a-stable* if

$$\mathrm{Stab_a}(y) := \{\, T \in \mathcal{P}_{\text{fin}} T_F(A) \mid f_T(y(T)) = a \,\} \in \mathcal{U}$$

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Lemma

There exists a mapping $\_^\bullet\colon A \to \prod_T Y_T$ such that $a^\bullet$ is a-stable.

We can define $a^\bullet \in \prod_T Y_T$ such that

$$a^\bullet(T) = \begin{cases} (f_T|_{Y_T})^{-1}(a) & \text{if } a \in \operatorname{Var} T, \\ y_T & \text{if } a \notin \operatorname{Var} T. \end{cases}$$

Lemma

Let $x, y \in \prod_T Y_T$. If x, y are a-stable, then $[\![x = y]\!] \in \mathcal{U}$.

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Let Y be the set of all a-stable elements for some $a \in A$. Than $Y \subseteq \prod_T Y_T \subseteq \prod_T B_T$ and so $[Y/\mathcal{U}] \leq \prod_T B_T/\mathcal{U}$. Which implies $[Y/\mathcal{U}] \in \mathrm{SP_U}(\mathcal{K})$.

Due to the lemmmata the mapping $g\colon A \to Y/\mathcal{U}$ such that $a \mapsto a^\bullet/\mathcal{U}$ is a bijection.

A mapping $f\colon [Y/\mathcal{U}] \to \mathcal{A}$ such that

$$f(t^{[Y/\mathcal{U}]}(a_1^\bullet/\mathcal{U}, \ldots, a_n^\bullet/\mathcal{U})) = t^{\mathcal{A}}(a_1, \ldots, a_n)$$

for any $t(a_1, \ldots, a_n) \in T_F(A)$ is a well defined homomorphism.

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Indeed, sets

$$M = [\![s^{\prod_T \mathcal{B}_T}(a_1^\bullet, \dots, a_n^\bullet) = t^{\prod_T \mathcal{B}_T}(a_1^\bullet, \dots, a_n^\bullet)]\!]$$

$$P = \overline{\{s(a_1, \dots, a_n), t(a_1, \dots, a_n)\}}$$

$$S = \bigcap_{i=1}^{n} \mathrm{Stab}_{a_i}(a_i^\bullet)$$

are elements of $\mathcal{U}$. And using $T \in M \cap P \cap S$ we can prove

$$s^{[Y/\mathcal{U}]}(a_1^\bullet/\mathcal{U}, \dots, a_n^\bullet/\mathcal{U}) = t^{[Y/\mathcal{U}]}(a_1^\bullet/\mathcal{U}, \dots, a_n^\bullet/\mathcal{U})$$

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Theorem

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Sketch of proof: There exist algebras $\mathcal{B}_i \in K$ for $i \in I$, an ultrafilter $\mathcal{U} \subseteq \mathcal{P}(I)$ and a homomorphism $h \colon \mathcal{B} \to \mathcal{A}$ such that $\mathcal{B} \leq \left(\prod_i \mathcal{B}_i\right)/\mathcal{U}$ and $\mathcal{A} = h(\mathcal{B})$.
Let us take an arbitrary finite set $T \in T_F(A)$.

- For every $a \in \operatorname{Var} T$ let us take a fixed element $a' \in \mathcal{B}$ such that $h(a') = a$.

$$Y_1 = \{\, a' \in \mathcal{B} \mid a \in \operatorname{Var} T \,\} \cup \{\, t^{\mathcal{B}}(a'_1, \ldots, a'_n) \in \mathcal{B} \mid t(a_1, \ldots, a_n) \in T \,\}$$

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- For every $b \in Y_1$ let us take a fixed $v(b) \in \prod_i \mathcal{B}_i$ such that $v(b)/\mathcal{U} = b$.

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$$[\![v(t^{(\prod_i \mathcal{B}_i)/\mathcal{U}}(a_1', \ldots, a_n')) = t^{\prod_i \mathcal{B}_i}(v(a_1'), \ldots, v(a_n'))]\!] \in \mathcal{U}.$$

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Finally, we prove $\mathcal{B}_j$, Y and f satisfy the conditions of the definition of FCP.

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Equivalent formulation of Jónsson's lemma:

Corollary

Let $\mathcal{K}$ be a class of algebras of the same type such that $\mathcal{V}(\mathcal{K})$ is a congruence distributive variety. If $\mathcal{A} \in \mathcal{V}(\mathcal{K})$ is subdirectly irreducible then $\mathcal{A}$ satisfies the finite coverability property for the class $\mathcal{K}$.

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References

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Thank you for your attention!