A new characterization of the class $\text{HSP}_U(\mathcal{K})$

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Finite Embeddability Property

Definition

A class of algebras $\mathcal{K}$ has the finite embeddability property (FEP) if every finite partial subalgebra of any algebra from $\mathcal{K}$ can be embedded into a finite member of $\mathcal{K}$.

- useful when dealing with the word problem
- applications in logic
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A different approach to FEP

**Definition**

An algebra \( \mathcal{A} \) satisfies the generalized finite embeddability property (GFEP) for a class \( \mathcal{K} \) of algebras of the same type if every finite partial subalgebra of \( \mathcal{A} \) can be embedded into an algebra from \( \mathcal{K} \).

**Theorem**

An algebra \( \mathcal{A} \) satisfies GFEP for \( \mathcal{K} \) if and only if \( \mathcal{A} \in \text{ISP}_U(\mathcal{K}) \).

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Finite Coverability Property

Definition

Let $\mathcal{A} = (A, F)$ be an algebra and $\mathcal{K}$ be a class of algebras of the type $F$. We say $\mathcal{A}$ satisfies the finite coverability property for the class $\mathcal{K}$ if for every finite set of terms $T \subseteq T_F(A)$ there exist an algebra $B \in \mathcal{K}$, a mapping $f : B \rightarrow A$ and a set $Y \subseteq B$ such that

- $f|_Y : Y \rightarrow \text{Var } T$ is a bijection,
- if $t(a_1, \ldots, a_n) \in T$ and $y_1, \ldots, y_n \in Y$ are such that $fy_i = a_i$ then
  
  $$ft^B(y_1, \ldots, y_n) = t^A(a_1, \ldots, a_n).$$

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Theorem

Let $A$ satisfy the finite coverability property for the class $\mathcal{K}$ then $A \in \text{HSP}_U(\mathcal{K})$.

Sketch of proof: There exists an ultrafilter $U$ on the set $\mathcal{P}_{\text{fin}} T_F(A)$ such that it contains all sets

$$\overline{T} = \{T' \in \mathcal{P}_{\text{fin}} T_F(A) \mid T \subseteq T'\},$$

where $T \in \mathcal{P}_{\text{fin}} T_F(A)$.

For every $T \in \mathcal{P}_{\text{fin}} T_F(A)$ there exist an algebra $B_T \in \mathcal{K}$, a mapping $f_T : B_T \to A$ and a set $Y_T \subseteq B_T$ satisfying the conditions of FCP.

Let $a \in A$ then $y \in \prod_T Y_T$ is called $a$-stable if

$$\text{Stab}_a(y) := \{T \in \mathcal{P}_{\text{fin}} T_F(A) \mid f_T(y(T)) = a\} \in U.$$
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Lemma

There exists a mapping $\cdot : A \to \prod_T Y_T$ such that $a^\cdot$ is $a$-stable.

We can define $a^\cdot \in \prod_T Y_T$ such that

$$a^\cdot(T) = \begin{cases} (f_T|_{Y_T})^{-1}(a) & \text{if } a \in \text{Var } T, \\ y_T & \text{if } a \notin \text{Var } T. \end{cases}$$

Lemma

Let $x, y \in \prod_T Y_T$. If $x, y$ are $a$-stable, then $[x = y] \in \mathcal{U}$.

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Let $x, y \in \prod_T Y_T$ be such that $[x = y] \in \mathcal{U}$. If $x$ is $a$-stable then also $y$ is $a$-stable.
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Let $Y$ be the set of all $a$-stable elements for some $a \in A$. Then $Y \subseteq \prod_T Y_T \subseteq \prod_T B_T$ and so $[Y/U] \leq \prod_T B_T/U$. Which implies $[Y/U] \in \text{SP}_U(K)$.

Due to the lemmmata the mapping $g: A \to Y/U$ such that $a \mapsto a^* / U$ is a bijection.

A mapping $f: [Y/U] \to A$ such that

$$f(t^{[Y/U]}(a_1^*/U, \ldots, a_n^*/U)) = t^A(a_1, \ldots, a_n)$$

for any $t(a_1, \ldots, a_n) \in T_F(A)$ is a well defined homomorphism.
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for any $t(a_1, \ldots, a_n) \in T_F(A)$ is a well defined homomorphism.
Indeed, sets

\[ M = \left\{ s^{\prod_{T}B_{T}}(a_{1}, \ldots, a_{n}) = t^{\prod_{T}B_{T}}(a_{1}, \ldots, a_{n}) \right\} \]

\[ P = \left\{ s(a_{1}, \ldots, a_{n}), t(a_{1}, \ldots, a_{n}) \right\} \]

\[ S = \bigcap_{i=1}^{n} \text{Stab}_{a_{i}}(a_{i}^{\bullet}) \]

are elements of \( \mathcal{U} \). And using \( T \in M \cap P \cap S \) we can prove

\[ s^{\left[Y/\mathcal{U}\right]}(a_{1}/\mathcal{U}, \ldots, a_{n}/\mathcal{U}) = t^{\left[Y/\mathcal{U}\right]}(a_{1}/\mathcal{U}, \ldots, a_{n}/\mathcal{U}) \]

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Theorem

Let $\mathcal{A} = (A, F)$ be an algebra and let $\mathcal{K}$ be a class of algebras of the type $F$. If $\mathcal{A} \in \text{HSP}_U(\mathcal{K})$ then $\mathcal{A}$ satisfies the finite coverability property for the class $\mathcal{K}$.

Sketch of proof: There exist algebras $B_i \in \mathcal{K}$ for $i \in I$, an ultrafilter $U \subseteq \mathcal{P}(I)$ and a homomorphism $h: B \to A$ such that $B \leq (\prod_i B_i) / U$ and $A = h(B)$.

Let us take an arbitrary finite set $T \in T_F(A)$.

- For every $a \in \text{Var } T$ let us take a fixed element $a' \in B$ such that $h(a') = a$.

$$Y_1 = \{ a' \in B \mid a \in \text{Var } T \} \cup \{ t^B(a'_1, \ldots, a'_n) \in B \mid t(a_1, \ldots, a_n) \in T \}$$
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For any $a_1, \ldots, a_n \in \text{Var } T$ and $t \in T$ we prove

\[ \left[ v(t(\prod_i B_i)/\mathcal{U}(a_1', \ldots, a_n')) = t\prod_i B_i(v(a_1'), \ldots, v(a_n')) \right] \in \mathcal{U}. \]

We can prove $W = W_1 \cap W_2 \in \mathcal{U}$ where

\[ W_1 = \bigcap_{t(a_1,\ldots,a_n) \in T} \left[ v(t \ldots) = t(v \ldots) \right] \]

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Equivalent formulation of Jónsson’s lemma:

Corollary

Let $\mathcal{K}$ be a class of algebras of the same type such that $\mathcal{V}(\mathcal{K})$ is a congruence distributive variety. If $\mathcal{A} \in \mathcal{V}(\mathcal{K})$ is subdirectly irreducible then $\mathcal{A}$ satisfies the finite coverability property for the class $\mathcal{K}$.
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References


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