# Algorithmic Sahlqvist Preservation for Modal Compact Hausdorff Spaces

Zhiguang Zhao

Delft University of Technology, Delft, the Netherlands

TACL, Prague, 28th Jun, 2017

# Motivation and Aim

$$
\begin{tikzcd}
\mathbb{CABAO} \arrow[r, leftrightarrow, "\text{Discrete duality}"] & \mathbb{KF} \\
\mathbb{BAO} \arrow[u, hook, "(\cdot)^{\delta}"] \arrow[r, leftrightarrow, "\text{Goldblatt/ Stone duality}"] & \mathbb{DGF} \arrow[u, "\mathsf{U}"']
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\mathbb{CABAO} \arrow[r, leftrightarrow, "\text{Discrete duality}"] & \mathbb{KF} \\
\mathbb{MCR} \arrow[u, hook, "(\cdot)_{\mathcal{T}}"] \arrow[r, leftrightarrow, "\text{M-Isbell duality}"] & \mathbb{MCH} \arrow[u, "\mathsf{U}"']
\end{tikzcd}
$$

# Motivation and Aim

$$
\begin{tikzcd}
\mathbb{CABAO} \arrow[rr, leftrightarrow, "\text{Discrete duality}"] & & \mathbb{KF} \\
\\
\mathbb{BAO} \arrow[uu, hook, "(\cdot)^{\delta}"] \arrow[rr, leftrightarrow, "\text{Goldblatt/ Stone duality}"] & & \mathbb{DGF} \arrow[uu, "\mathsf{U}"']
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
\mathbb{CABAO} \arrow[rr, leftrightarrow, "\text{Discrete duality}"] & & \mathbb{KF} \\
\\
\mathbb{MCR} \arrow[uu, hook, "{\color{red}(\cdot)_{\mathcal{T}}}"] \arrow[rr, leftrightarrow, "\text{M-Isbell duality}"] & & \mathbb{MCH} \arrow[uu, "\mathsf{U}"']
\end{tikzcd}
$$

Which inequalities are preserved by the embedding $(\cdot)_{\mathcal{T}}$?


G. Bezhanishvili, N. Bezhanishvili, and J. Harding. "Modal compact Hausdorff spaces". *JLC*, 25(1):1–35, 2015.

# Motivation and Aim

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathbb{CABAO} \arrow[r, leftrightarrow, "\text{Discrete duality}"] & \mathbb{KF} \\
\mathbb{BAO} \arrow[u, hook, "(\cdot)^{\delta}"] \arrow[r, leftrightarrow, "\text{Goldblatt/ Stone duality}"] & \mathbb{DGF} \arrow[u, "\mathsf{U}"']
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathbb{CABAO} \arrow[r, leftrightarrow, "\text{Discrete duality}"] & \mathbb{KF} \\
\mathbb{MCR} \arrow[u, hook, "{\color{red}(\cdot)_{\mathcal{T}}}"] \arrow[r, leftrightarrow, "\text{M-Isbell duality}"] & \mathbb{MCH} \arrow[u, "\mathsf{U}"']
\end{tikzcd}
$$

Which inequalities are preserved by the embedding $(\cdot)_{\mathcal{T}}$?

G. Bezhanishvili, N. Bezhanishvili, and J. Harding. “Modal compact Hausdorff spaces”. *JLC*, 25(1):1–35, 2015.

We explore this question with the **ALBA** technology

# The ALBA technology

- Calculus/Algorithm which computes the first-order correspondent of input formulas/inequalities;
- Based on the order-theoretic properties of the algebraic interpretation of logical connectives;
- Guaranteed to succeed on **inductive formulas/inequalities**;
- Inductive formulas/inequalities generalise Sahlqvist formulas/inequalities;
- General definition for arbitrary normal and regular LEs, based on the order-theoretic properties of the algebraic interpretation of logical connectives.

## Standard proof-strategy for canonicity via ALBA

$$\begin{tikzcd}
\mathbb{A} \models \alpha \leq \beta \arrow[d, Leftrightarrow] & \mathbb{A}^\delta \models \alpha \leq \beta \arrow[dd, Leftrightarrow] \\
\mathbb{A}^\delta \models_{\mathbb{A}} \alpha \leq \beta \arrow[d, Leftrightarrow] & \\
\mathbb{A}^\delta \models_{\mathbb{A}} \mathsf{ALBA}(\alpha \leq \beta) \arrow[r, Leftrightarrow] & \mathbb{A}^\delta \models \mathsf{ALBA}(\alpha \leq \beta)
\end{tikzcd}$$

Let us generalise this strategy to modal compact Hausdorff spaces.

## U-shaped Argument

$\mathcal{T}$: modal compact Hausdorff space;
$\mathbb{L}_{\mathcal{T}}$: the modal compact regular frame associated with $\mathcal{T}$;
$\mathbb{F}_{\mathcal{T}}$: the underlying Kripke frame of $\mathcal{T}$;
$\mathbb{B}_{\mathbb{F}_{\mathcal{T}}}$: the complex algebra of $\mathbb{F}_{\mathcal{T}}$.

$$
\begin{array}{ccc}
\mathbb{L}_{\mathcal{T}} \models \phi \leq \psi & & \mathbb{B}_{\mathbb{F}_{\mathcal{T}}} \models \phi \leq \psi \\
\Updownarrow & & \\
\mathbb{B}_{\mathbb{F}_{\mathcal{T}}} \models_{\mathbb{L}_{\mathcal{T}}} \phi \leq \psi & & \Updownarrow \\
\Updownarrow & & \\
\mathbb{B}_{\mathbb{F}_{\mathcal{T}}} \models_{\mathbb{L}_{\mathcal{T}}} \mathrm{Pure}(\phi \leq \psi) & \Leftrightarrow & \mathbb{B}_{\mathbb{F}_{\mathcal{T}}} \models \mathrm{Pure}(\phi \leq \psi)
\end{array}
$$

To implement this strategy, we need to adapt ALBA to the modal compact Hausdorff setting.

# Main Results

- ALBA adapted to the modal compact Hausdorff setting (**MH-ALBA**);
- Rules of MH-ALBA **sound** w.r.t. **open** assignments (i.e. the assignments mapping into $\mathbb{L}_{\mathcal{T}}$);
- MH-ALBA succeeds on **1-Sahlqvist inequalities** and $\partial$-**Sahlqvist inequalities**;
- Corollary: 1-Sahlqvist and $\partial$-Sahlqvist inequalitie are **Hausdorff-canonical**.

# Main Results

Prop: set of propositional variables,

$$\mathcal{L} \ni \varphi ::= p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box\varphi \mid \Diamond\varphi.$$

## Definition (1-Sahlqvist inequalities)

$\phi \leq \psi$ is 1-*Sahlqvist* if $\phi = \phi'(\chi_1/z_1, \ldots \chi_n/z_n)$ such that

1. $\phi'(z_1, \ldots, z_n)$ is built out of $\wedge, \vee, \Diamond$;
2. every $\chi$ is of the form $\Box^n p$, $\Box^n \top$, $\Box^n \bot$ for $n \geq 0$;
3. $\psi$ is a formula in the positive modal language.

The $\partial$-*Sahlqvist inequality* $\phi \leq \psi$ where $\psi = \psi'(\chi_1/z_1, \ldots \chi_n/z_n)$ is defined dually.

# Future Work

- Extensions to arbitrary modal signatures and fixed-point operators;
- Extensions to inductive formulas;
- Studying the "Hausdorff-canonical extensions" of modal compact Hausdorff spaces purely algebraically.