$\aleph_1(=\omega_1)$ and the modal μ-calculus[1]

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Plan

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The modal μ-calculus

- Formulas of L_μ :

$$\phi := x \mid y \mid \neg y \mid \top \mid \phi \wedge \phi \mid \bot \mid \phi \vee \phi \mid \langle a \rangle \phi \mid [a]\phi$$
$$\mid \mu_x.\phi \mid \nu_x.\phi\,.$$

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  and define

$$[\![\mu_x.\phi]\!]_{\mathcal{M}} := \text{least fixed-point of } \phi_{\mathcal{M},x} \,,$$
$$[\![\nu_x.\phi]\!]_{\mathcal{M}} := \text{greatest fixed-point of } \phi_{\mathcal{M},x} \,.$$

Closure ordinals

Put

$$\phi^{\alpha}_{\mathcal{M},x}(\emptyset) := \bigcup_{\beta<\alpha} \phi^{\beta}_{\mathcal{M},x}(\emptyset)\,, \qquad \phi^{\alpha+1}_{\mathcal{M},x}(\emptyset) := \phi_{\mathcal{M},x}(\phi^{\alpha}_{\mathcal{M},x}(\emptyset))\,.$$

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Problem. Given $\phi \in \mathsf{L}_{\mu}$, does there exist an ordinal α such that

$$[\![\mu_x.\phi]\!]_{\mathcal{M}} = \phi^{\alpha}_{\mathcal{M},x}(\emptyset)\,, \qquad \textbf{for each model } \mathcal{M}?$$

If so, call the least such ordinal the *closure ordinal of* $\phi(x)$.

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- What about closure ordinals of formulas in fragments of L_μ?
- ... and on restricted classes of models and/or modal varieties?

Examples, known (?) results

- $\langle\ \rangle x$ has closure ordinal 0.
- $y \vee \langle\ \rangle x$ has closure ordinal ω,
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  then it has a closure ordinal $\leq \omega$.
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Theorem (Afshari and Leigh, 2013). If $\phi(x) \in \mathsf{L}_\mu$ has a closure ordinal α and no greatest fixed-point, then $\alpha < \omega^2$.

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κ-continuous functions and their least fixed-points

Let κ be a regular cardinal.

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Definition.

- A family $\mathcal{F} \subseteq P(A)$ is κ-*directed* if any subfamily of $\mathcal{F}$ of cardinality $< \kappa$ has an upper bound in $\mathcal{F}$.
- A monotone function $f : P(A) \longrightarrow P(A)$ is κ-*continuous* if it preserves unions of κ-directed families.

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Theorem (Knaster, Tarski, Kleene, . . . , Cousot-Cousot, . . . , ?, G.S.). If f is κ-continuous, then

$$\mathtt{lfp}.f = f^{\kappa}(\emptyset)\,.$$

Stability under fixed-points

The following statements appeared in (S. 2002), for accessible functors and their initial(terminal) (co)algebras.

Proposition.

- If $f(x, y)$ is κ-continuous in x and y and $\kappa \geq \aleph_0$,
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- If $f(x, y)$ is κ-continuous in x and y and $\kappa \geq \aleph_1$,
  then $\nu_x.f(x, y)$ κ-continuous in y.

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Continuity, for formulas

Definition. A formula $\phi(x) \in \mathsf{L}_\mu$ is κ-continuous if, for each model $\mathcal{M}$, the function $\phi_{\mathcal{M},x}$ is κ-continuous.

If a formula is κ-continuous,
then it has a closure ordinal with κ as upper bound.

The $\mathcal{C}_0(X)$-fragment

The fragment $\mathcal{C}_0(X)$ is generated by the following grammar:

$$\phi := x \mid y \mid \neg y \mid \top \mid \phi \wedge \phi \mid \bot \mid \phi \vee \phi \mid \langle a \rangle \phi \\ \mid \mu_z.\psi \,,$$

where $x \in X$ and $\psi \in \mathcal{C}_0(X \cup \{ z \})$.

Theorem (Fontaine 2008). For every $\phi \in \mathcal{C}_0(X)$ and each variable $x \in X$, the formula $\phi(x)$ is $\aleph_0$-continuous.

In particular each $\phi(x) \in \mathcal{C}_0(x)$ has closure ordinal $\leq \omega$.

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In particular each $\phi(x) \in \mathcal{C}_1(x)$ has a closure ordinal $\leq \omega_1$.

Decidability

Theorem (Fontaine 2008). For each formula $\phi(x)$ there is a formula $\psi(x) \in \mathcal{C}_0(x)$ such that TFAE:

- $\phi(x)$ is $\aleph_0$-continuous,
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Theorem (G.S. 2017). For each formula $\phi(x)$ we can construct a formula $\phi^\flat(x) \in \mathcal{C}_1(x)$ such that the follwing are equivalent:

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Corollary. It is decidable whether a formula $\phi(x) \in \mathsf{L}_\mu$ is κ-continuous, for some regular cardinal κ.

Other κ-continuous fragments?

Corollary. If a formula $\phi(x) \in \mathsf{L}_\mu$ is κ-continuous, for some regular cardinal κ, then it is $\aleph_1$-continuous.

Corollary. $\aleph_0$ and $\aleph_1$ are the only regular cardinal of interest to the modal μ-calculus.

Namely, $\mathcal{C}_0(x)$ and $\mathcal{C}_1(x)$ are the only fragments of the modal μ-calculus determined by κ-continuity at some regular cardinal κ.

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Proof. We have

$$s \models [v]\bot \vee \nu_z.(\langle v\rangle x \wedge \langle h\rangle z)$$

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either there no vertical steps from s,
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The above formula belongs to $\mathcal{C}_1(x)$. A (bimodal) model $\mathcal{M}$ on the set

$$\omega \times \omega_1 = \{ (n, \alpha) \mid 0 \leq n < \omega, \alpha < \omega_1 \},$$

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Finally, we translate the bimodal formula above into a monomodal one (and similarly for $\mathcal{M}$).

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Theorem (G. S. 2017). Every formal expression built up from $1, \omega, \omega_1$ using $+$ gives rise to a closure ordinal of the modal μ-calculus.

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- Semantic tools to characterize the spectrum of ordinals of L_μ (see also Gaëlle and Yde's preprint).
- Yet, the spectrum of cardinals of L_μ has been fully characterized.

Thanks ! Questions ?

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