Varieties of De Morgan Monoids II: Covers of Atoms

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De Morgan monoids

A De Morgan monoid $\boldsymbol{A} = \langle A; \vee, \wedge, \cdot, \neg, t\rangle$ comprises

- a distributive lattice $\langle A; \vee, \wedge\rangle$,
- a square-increasing ($x \leq x \cdot x$) commutative monoid $\langle A; \cdot, t\rangle$,
- satisfying $x = \neg\neg x$
- and $x \cdot y \leq z$ iff $x \cdot \neg z \leq \neg y$.
- $x \to y := \neg(x \cdot \neg y)$

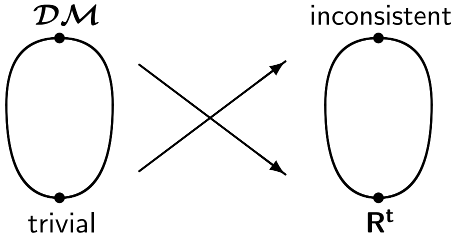
$\mathcal{DM}$ denotes the variety of all De Morgan monoids.

Algebraic logic

The logic $\mathbf{R}^\mathsf{t}$ can be characterized as follows

$$\gamma_1, \ldots, \gamma_n \vdash_{\mathbf{R}^\mathsf{t}} \alpha \ \text{ iff } \ \mathcal{DM} \vDash \big(t \le \gamma_1 \,\&\, \ldots \,\&\, t \le \gamma_n\big) \Rightarrow t \le \alpha.$$

$\mathsf{L}_{\mathbb{V}}(\mathcal{DM})$:

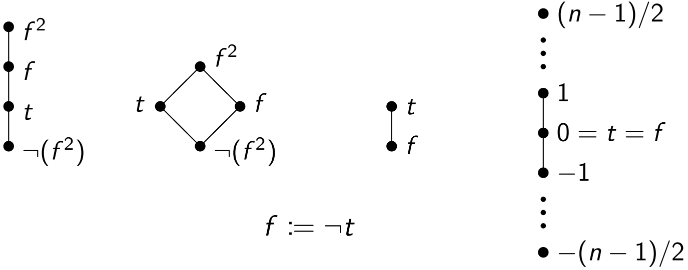


Subvarieties of $\mathcal{DM}$

Axiomatic extensions of $\mathbf{R}^\mathsf{t}$

Important algebras

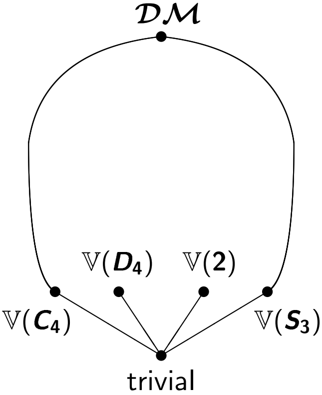
$\boldsymbol{C_4}$ $\boldsymbol{D_4}$ $\mathbf{2}$ $\boldsymbol{S_n}$ (n odd)



- The first three are exactly the simple 0-generated De Morgan monoids, see Slaney (1989).
- For any positive odd number n, the $\cdot$ of $\boldsymbol{S_n}$ is as follows:

$$\text{when } |i| \leq |j|, \text{ then } i \cdot j = \begin{cases} j & \text{if } |i| \neq |j| \\ i \wedge j & \text{otherwise.} \end{cases}$$

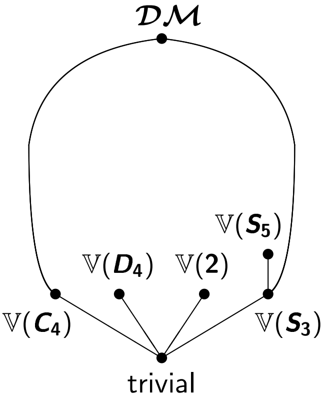
Atoms of $\mathrm{L}_{\mathbb{V}}(\mathcal{DM})$



Subvarieties of $\mathcal{DM}$

We investigate the covers of the atoms in $\mathrm{L}_{\mathbb{V}}(\mathcal{DM})$.

Covers of $\mathbb{V}(\mathbf{2})$ and $\mathbb{V}(\boldsymbol{S_3})$



Subvarieties of $\mathcal{DM}$

- The join of any two atoms is a cover of both.
- The remaining covers are precisely the *join-irreducible* (JI) covers.

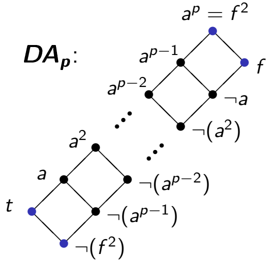
Thm.

- $\mathbb{V}(\mathbf{2})$ *has no JI cover.*
- *The only JI cover of* $\mathbb{V}(\boldsymbol{S_3})$ *is* $\mathbb{V}(\boldsymbol{S_5})$.

Covers of $\mathbb{V}(\boldsymbol{D_4})$

Thm. *Every join-irreducible cover of $\mathbb{V}(\boldsymbol{D_4})$ has the form $\mathbb{V}(\boldsymbol{A})$ for some simple 1-generated De Morgan monoid $\boldsymbol{A}$, where $\boldsymbol{D_4}$ embeds into $\boldsymbol{A}$ but is not isomorphic to $\boldsymbol{A}$.*

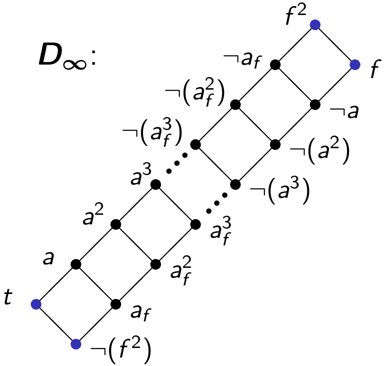
$\boldsymbol{DA_p}$:



- For every prime p, the algebra $\boldsymbol{DA_p}$ generates a cover of $\mathbb{V}(\boldsymbol{D_4})$,
- so there are infinitely many covers of $\mathbb{V}(\boldsymbol{D_4})$.

A non-finitely generated cover of $\mathbb{V}(\boldsymbol{D_4})$

$\boldsymbol{D_\infty}$:



- Not all covers of $\mathbb{V}(\boldsymbol{D_4})$ are finitely generated,
- for example, $\boldsymbol{D_\infty}$ generates a cover of $\mathbb{V}(\boldsymbol{D_4})$ that is not finitely generated.

Covers of $\mathbb{V}(\boldsymbol{C_4})$

More cases, as $\boldsymbol{C_4}$ has diverse homomorphic pre-images. In fact:

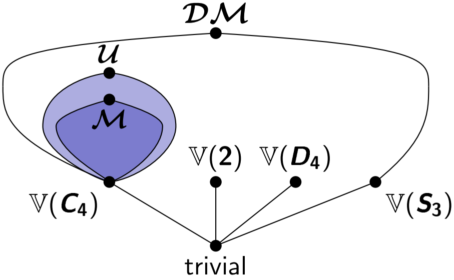
Thm. (Slaney) *If $h : \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism from a finitely subdirectly irreducible De Morgan monoid into a 0-generated De Morgan monoid, then h is an isomorphism or $\boldsymbol{B} \cong \boldsymbol{C_4}$.*

- There is a largest subvariety $\boldsymbol{\mathcal{U}}$ of $\boldsymbol{\mathcal{DM}}$ such that every non-trivial member of $\boldsymbol{\mathcal{U}}$ has $\boldsymbol{C_4}$ as a homomorphic image.
- $\boldsymbol{\mathcal{U}}$ is finitely axiomatized.
- There is a largest subvariety $\boldsymbol{\mathcal{M}}$ of $\boldsymbol{\mathcal{DM}}$ such that $\boldsymbol{C_4}$ is a retract of all non-trivial members of $\boldsymbol{\mathcal{M}}$.
- $\boldsymbol{\mathcal{M}}$ is axiomatized, relative to $\boldsymbol{\mathcal{U}}$, by $t \leq f$.

Covers of $\mathbb{V}(\boldsymbol{C_4})$

Thm. *If $\mathcal{K}$ is a join-irreducible cover of $\mathbb{V}(\boldsymbol{C_4})$, then exactly one of the following holds.*

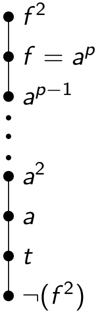
1. $\mathcal{K} = \mathbb{V}(\boldsymbol{A})$ *for some simple 1-generated De Morgan monoid* $\boldsymbol{A}$, *such that* $\boldsymbol{C_4}$ *embeds into* $\boldsymbol{A}$ *but is not isomorphic to* $\boldsymbol{A}$.
2. $\mathcal{K} = \mathbb{V}(\boldsymbol{A})$ *for some (finite) 0-generated subdirectly irreducible De Morgan monoid* $\boldsymbol{A} \in \mathcal{U} \setminus \mathcal{M}$.
3. $\mathcal{K} \subseteq \mathcal{M}$.



Condition 1

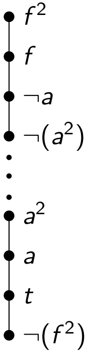
1. $\mathcal{K} = \mathbb{V}(\boldsymbol{A})$ for some simple 1-generated De Morgan monoid $\boldsymbol{A}$, such that $\boldsymbol{C_4}$ embeds into $\boldsymbol{A}$ but is not isomorphic to $\boldsymbol{A}$.

$\boldsymbol{A_p}$:



- For every prime p, the algebra $\boldsymbol{A_p}$ generates a cover of $\mathbb{V}(\boldsymbol{C_4})$,
- so, there are infinitely many covers of $\mathbb{V}(\boldsymbol{C_4})$ that satisfy condition 1.

$\boldsymbol{A_\infty}$:



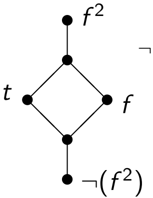
- There are covers of $\mathbb{V}(\boldsymbol{C_4})$ that are not finitely generated,
- for example, $\boldsymbol{A_\infty}$ generates a cover of $\mathbb{V}(\boldsymbol{C_4})$.

Condition 2

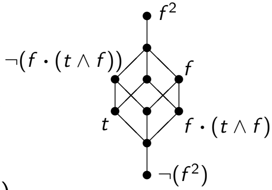
2. $\mathcal{K} = \mathbb{V}(\boldsymbol{A})$ for some (finite) 0-generated subdirectly irreducible De Morgan monoid $\boldsymbol{A} \in \mathcal{U} \setminus \mathcal{M}$.

Slaney (1989) characterized all the 0-generated subdirectly irreducible De Morgan monoids. They are all finite, and apart from the simple ones, they are:

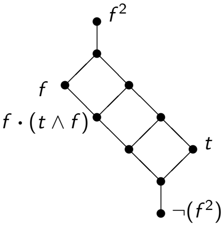
$\boldsymbol{C_5}$:



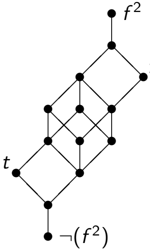
$\boldsymbol{C_6}$:



$\boldsymbol{C_7}$:



$\boldsymbol{C_8}$:



Condition 3

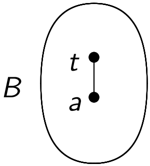
3. $\mathcal{K} \subseteq \mathcal{M}$

Every subdirectly irreducible algebra in $\mathcal{M}$ arises by a construction of Slaney (1993) from a **Dunn monoid** $\boldsymbol{B}$ [essentially a De Morgan monoid without the involution $\neg$], i.e.,

> *a square-increasing distributive lattice-ordered commutative monoid* $\langle B; \vee, \wedge, \cdot, \rightarrow, t\rangle$ *that satisfies the law of residuation*
>
> $$x \leq y \rightarrow z \textit{ iff } x \cdot y \leq z.$$

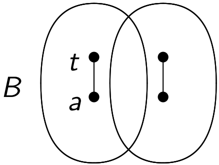
Let's call this construction **skew reflection**.

Skew Reflection



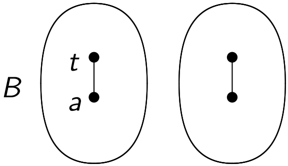
Dunn monoid

Skew Reflection



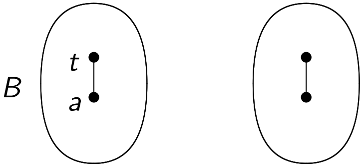
Dunn monoid

Skew Reflection



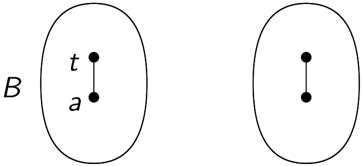
Dunn monoid

Skew Reflection



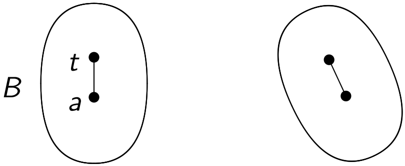
Dunn monoid

Skew Reflection



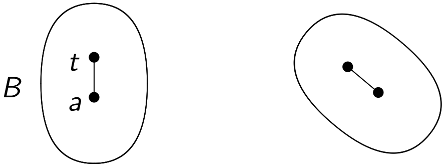
Dunn monoid

Skew Reflection



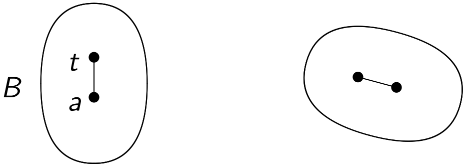
Dunn monoid

Skew Reflection



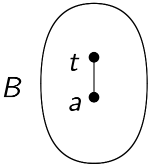
Dunn monoid

Skew Reflection

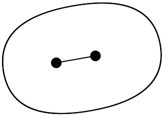


Dunn monoid

Skew Reflection



Dunn monoid



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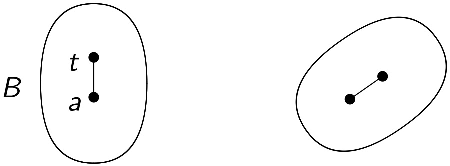

Dunn monoid

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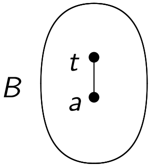
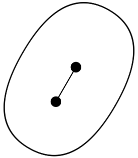


Dunn monoid



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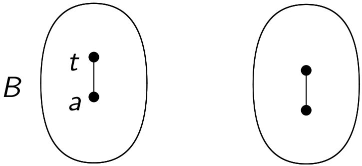
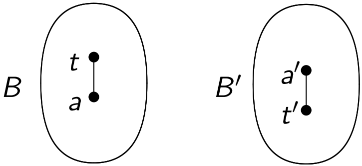


Dunn monoid

Skew Reflection



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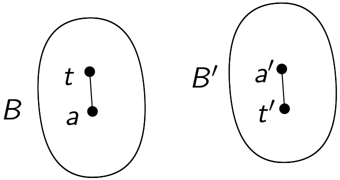

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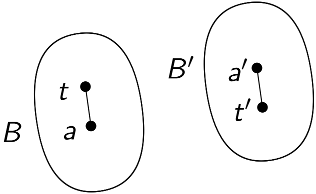

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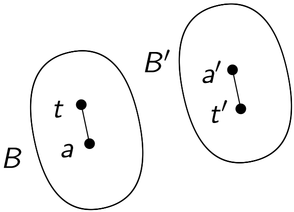

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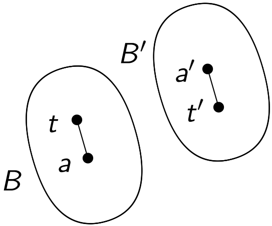

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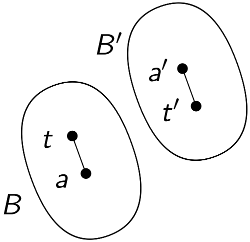

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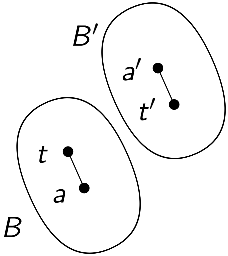

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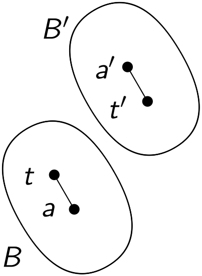

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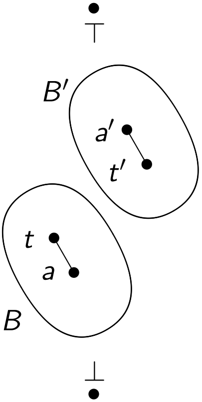

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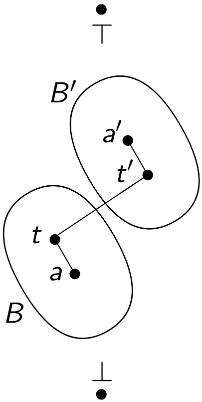


Declare that $a < b'$ for certain $a, b \in B$ in such a way that $\langle B \cup B' \cup \{\bot, \top\}; \leq\rangle$ is a distributive lattice, $t < t'$ and for all $a, b \in B$,

$$a < b' \text{ iff } t < (a \cdot b)'.$$

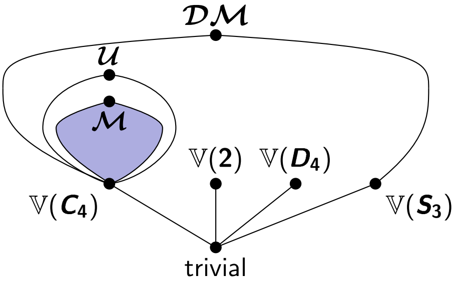
Then there is a unique way of turning the structure into a De Morgan monoid

$$S^{<}(\boldsymbol{B}) = \langle B \cup B' \cup \{\bot, \top\}; \vee, \wedge, \cdot, \neg, t\rangle \in \boldsymbol{\mathcal{M}},$$

of which $\boldsymbol{B}$ is a subreduct, where $\neg$ extends $'$. In particular, if we specify that $a < b'$ for all $a, b \in B$, then we get the **reflection** construction, which is an older idea, see Meyer (1973) and Galatos and Raftery (2004). In this case we write $R(\boldsymbol{B})$ for $S^{<}(\boldsymbol{B})$.

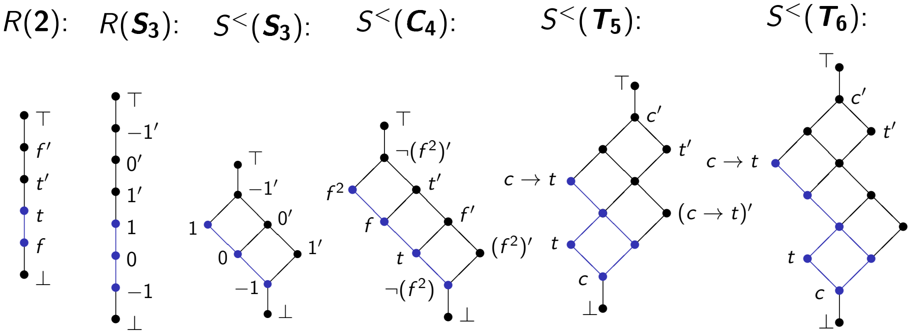
Covers of $\mathbb{V}(\boldsymbol{C_4})$ within $\boldsymbol{\mathcal{M}}$

Thm. *Let $\mathcal{K}$ be a cover of $\mathbb{V}(\boldsymbol{C_4})$ within $\boldsymbol{\mathcal{M}}$. Then $\mathcal{K} = \mathbb{V}(\boldsymbol{A})$ for some finite skew reflection $\boldsymbol{A}$ of a subdirectly irreducible Dunn monoid $\boldsymbol{B}$, where $\bot$ is meet-irreducible in $\boldsymbol{A}$, and $\boldsymbol{A}$ is generated by the greatest strict lower bound of t in $\boldsymbol{B}$.*



Covers of $\mathbb{V}(\boldsymbol{C_4})$ within $\mathcal{M}$

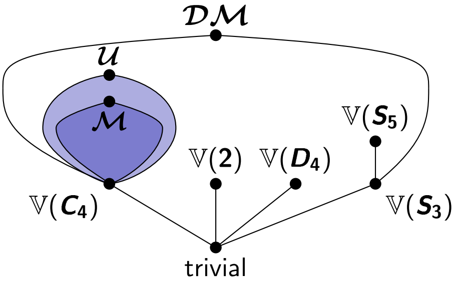
There are just six of these:



$\boldsymbol{T_5}$ is idempotent and $\boldsymbol{T_6}$ is idempotent except for $t' \wedge (c \to t)$.

Summary

Thm. *Every cover of* $\mathbb{V}(\boldsymbol{C_4})$ *within* $\boldsymbol{\mathcal{M}}$ *has no proper nontrivial subquasivariety other than* $\mathbb{V}(\boldsymbol{C_4})$.



Definitions

Atoms

Covers of $\mathbb{V}(\mathbf{2})$ and $\mathbb{V}(\boldsymbol{S_3})$

Covers of $\mathbb{V}(\boldsymbol{D_4})$

Covers of $\mathbb{V}(\boldsymbol{C_4})$

Skew Reflection

Covers of $\mathbb{V}(\boldsymbol{C_4})$ within $\mathcal{M}$