Varieties of De Morgan Monoids I : Minimality and Irreducible Algebras

T. Moraschini,[1] J.G. Raftery[2] and J.J. Wannenburg[2]

[1]Czech Academy of Sciences, Prague

[2]University of Pretoria, South Africa

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A **De Morgan monoid** $\boldsymbol{A} = \langle A; \cdot, \wedge, \vee, \neg, t\rangle$ comprises

- a distributive lattice $\langle A; \wedge, \vee\rangle$;
- a commutative monoid $\langle A; \cdot, t\rangle$ satisfying $x \leqslant x \cdot x$;
- an 'involution' $\neg \colon A \longrightarrow A$ satisfying $\neg\neg x = x$ and $x \cdot y \leqslant z \implies x \cdot \neg z \leqslant \neg y$ (so $\neg \colon \langle A; \wedge, \vee\rangle \cong \langle A; \vee, \wedge\rangle$).

Defining $x \to y = \neg(x \cdot \neg y)$ and $f = \neg t$, we obtain the

Law of Residuation: $x \cdot y \leqslant z \iff x \leqslant y \to z$;

and $\neg x = x \to f$.

$\mathcal{DM} = \{$all De Morgan monoids$\}$ is a variety.

It is congruence distributive, extensible and permutable.

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$$\vdash_{\mathbf{R^t}} \alpha \ \text{ ('}\alpha\text{ is a theorem of } \mathbf{R^t}\text{')} \ \text{ iff } \ \mathcal{DM} \models t \leqslant \alpha.$$

More generally, in the deducibility relation of the usual formal system for $\mathbf{R^t}$, we have $\gamma_1, \ldots, \gamma_n \vdash_{\mathbf{R^t}} \alpha$ iff

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There is a lattice anti-isomorphism from the extensions of $\mathbf{R^t}$ to the subquasivarieties of $\mathcal{DM}$, taking

axiomatic extensions onto subvarieties.

We study the latter as a route to the former.

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Relevance logic began in protest at 'paradoxes' of material implication, e.g., the weakening axiom $p \to (q \to p)$.

It has multiple interpretations, however, and now fits under the ideology-free umbrella of substructural logics.

Relative to these, $\mathbf{R^t}$ combines $\wedge, \vee$ distributivity with the contraction axiom $(p \to (p \to q)) \to (p \to q)$.

Urquhart (1984): $\mathbf{R^t}$ is undecidable.

Algebraic effects? Less explored—philosophical equivocation over the status of t: distinguished or not?

(In the absence of weakening, t is not equationally definable. The t–free reducts of De Morgan monoids don't form a variety, as they are not closed under subalgebras.
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Contraction amounts to the square-increasing law $x \leqslant x^2$ of $\mathcal{DM}$. Its effects include:

- Excluded middle: $t \leqslant x \vee \neg x$;
- Unique involution ($\neg$): if two algebras have the same $\cdot, \rightarrow, \wedge, \vee, t$ reduct, they are equal (Slaney, 2016).
- Algebras are simple iff t has just one strict lower bound.
- Finitely generated algebras are bounded.
  (If $\bot \leqslant x \leqslant \top$ for all x, then $\bot \cdot x = \bot$.)

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- Algebras are finitely subdirectly irreducible (FSI) iff t is join-prime: $t \leqslant x \vee y \implies t \leqslant x$ or $t \leqslant y$.

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Special features of De Morgan monoids:

- FSI bounded algebras are 'rigorously compact': if $\bot \leqslant x \leqslant \top$ for all x, then $\top \cdot x = \top$, unless $x = \bot$.
- $f^3 = f^2$.
- If a De Morgan monoid is 0–generated (i.e., it has no proper subalgebra), then it is finite (Slaney, 1980s).

  Just seven such algebras are subdirectly irreducible (SI).
- If $\boldsymbol{A} \in \mathcal{DM}$ is FSI, then $A = [t) \cup (f]$.

Here, we may have $t \leqslant f$. But if $f < t$, then t covers f.

Fact. In a De Morgan monoid, the demand $f \leqslant t$ is equivalent to idempotence of the whole algebra: $x^2 = x$ (for all x).

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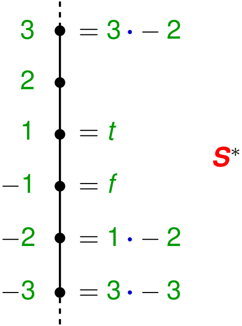

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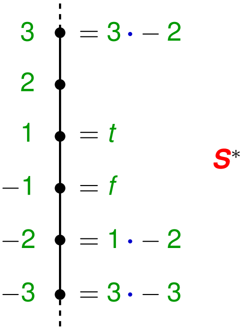

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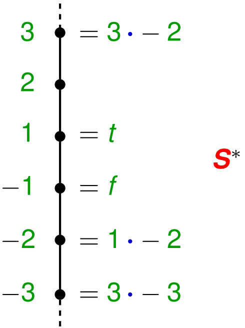

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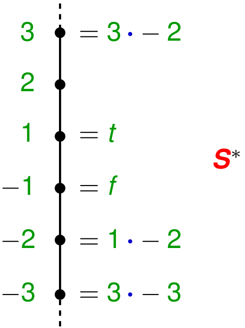

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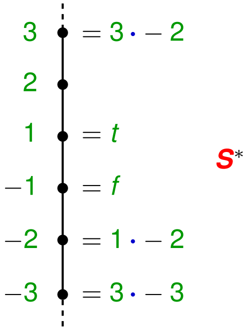

The n–element (unique) convex subalgebra of $\boldsymbol{S}^*$ or $\boldsymbol{S}$ is denoted by $\boldsymbol{S}_n$.

These are exactly the finitely generated SI Sugihara monoids (Dunn, 1970s), so $\mathcal{SM}$ is semilinear (i.e., Sugihara monoids are subdirect products of chains).

E.g., $\boldsymbol{S}_2$ is the Boolean algebra $-1 < 1$;

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$$[\neg(f^2), f^2] := \{x \in A : \neg(f^2) \leq x \leq f^2\}$$

and two chains of idempotent elements, $(\neg(f^2)]$ and $[f^2)$.

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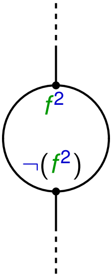

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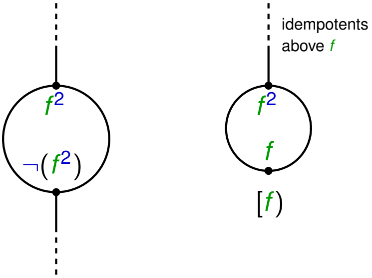

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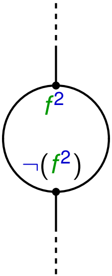

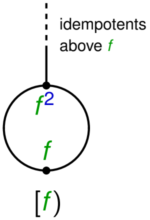

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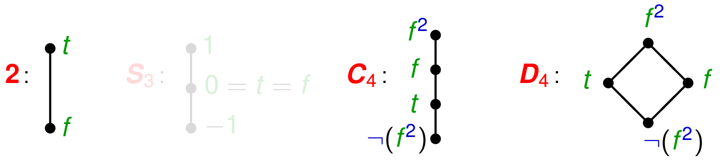

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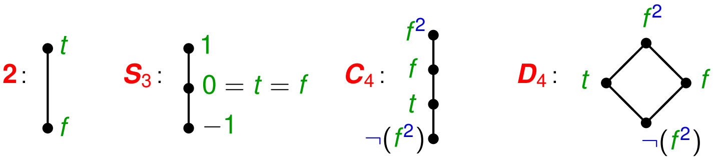

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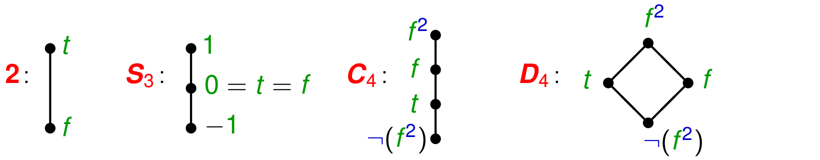

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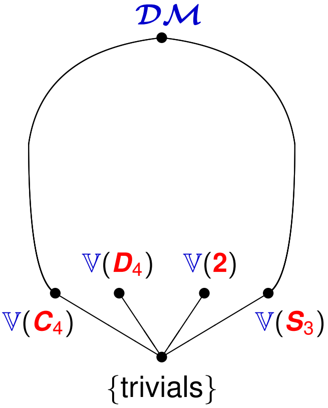

Subvarieties of $\mathcal{DM}$

On general grounds,

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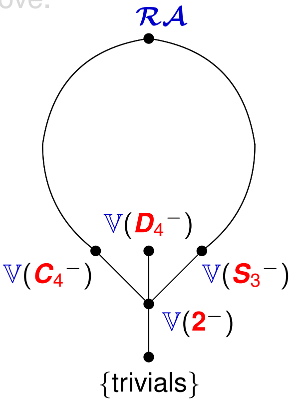

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