Modal logics over finite residuated lattices

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Topology, Algebra and Categories in Logic 2017, **Prague, Czech Republic**,

June 29, 2017

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  - local deduction, global over crisp frames
    Q2. (general) Global deduction
  - Q3. Is an axiomatization for the Global modal logic an axiomatization for the local one + $\frac{x \to y}{\Box x \to \Box y}$?
    (Q3'). Similar question restricting to crisp accessibility and adding $\frac{x}{\Box x}$

Preliminaries

- $\mathbf{A} = \langle A, \cdot, \rightarrow, \wedge, \vee, 0, 1 \rangle$ is a (bounded, commutative, integral) **residuated lattice** when
  - $\langle A, \wedge, \vee, 1, 0 \rangle$ is a bounded lattice (with order denoted $\leq$),
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  $e(v, \Box\varphi) = \bigwedge_{w \in W} \{Rvw \to e(w, \varphi)\} \qquad e(v, \Diamond\varphi) = \bigvee_{w \in W} \{Rvw \cdot e(w, \varphi)\}$
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- Same valid formulas.

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  Not known general interdefinability of modalities....
- Local classical modal logic enjoys DT $\Longrightarrow$ usually we say "modal logic" for the set of valid formulas or the global consequence.
  No longer (necessarily) true -nor even LDT.

Existing axiomatization

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(For SI residuated lattices (with a unique coatom), adding K and $\Box(x \vee \overline{c}) \rightarrow (\Box x \vee \overline{c})) \Longrightarrow$ completeness wrt. the no-$\Diamond$ fragment of $\Vdash^{l}_{C_{\mathbf{A}}}$.)

Both modal operators

answer to Q1

For $\mathcal{L} =$ (the previous A.S.)+ $\Box(x \to \overline{c}) \to (\Diamond x \to \overline{c})$, $\mathcal{L}$ is complete with respect to $\Vdash^{l}_{M_{\mathbf{A}(c)}}$ (also concerning only idempotent/crisp frames completeness).

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- Two RL equations concerning existing arbitrary infima/suprema are partially axiomatically represented

$$\bigwedge_{x \in X} x \to y = \bigvee X \to y \qquad \text{and} \quad \bigwedge_{x \in X} y \to x = y \to \bigwedge X$$

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It was not proven whether an axiomatic system for the global logic over $\mathbf{A}$/ $\mathbf{A}^{(c)}$ can be obtained by adding (the appropriated) necessity rule to an axiomatic system for the local one.

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$\mathcal{L} + N_\Box (= \frac{\varphi \to \psi}{\Box\varphi \to \Box\psi})$ is complete with respect to $\Vdash^{g}_{M_{\mathbf{A}^{(c)}}}$ (also considering the no-$\Diamond$ restriction and the idempotent cases)

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- Clear that $\Gamma \Vdash^l_{M_{\mathbf{A}^{(c)}}} \varphi \Longrightarrow \Gamma \Vdash^g_{M_{\mathbf{A}^{(c)}}} \varphi$.
- Thus soundness follows easily: $\Gamma \vdash_{\mathcal{L}+N_\Box} \varphi \Longrightarrow \Gamma \Vdash^g_{M_{\mathbf{A}^{(c)}}}$ (all axioms from $\mathcal{L}$ are sound in the global deduction, and so is $N_\Box$.)

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For the completeness direction, we will build appropriated canonical models.

On the canonical model(s)

Assume $\Gamma \nvdash_{\mathcal{L}+N_{\Box}} \varphi$. We define the Γ-canonical model by:

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2. $\Gamma \not\vdash_{\mathcal{L}+N_\Box} \varphi \implies C_{\mathcal{L}+N_\Box}(\Gamma) \not\models_{\mathbf{A}^{(c)}} \varphi$, so there is $h \in W$ for which $h(\varphi) < 1$.

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3. So this model would indeed serve to prove $\Gamma \not\Vdash^{g}_{M^{(c)}_{\mathbf{A}}} \varphi$.

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- $h(\Box\varphi) \stackrel{?}{=} \bigwedge_{g \in W} \{Rhg \to g(\varphi)\}$
  - $\leq$ direction is easy:

$$Rhg \to g(\varphi) =$$
$$\bigwedge_{\psi \in MFm} \{((h(\Box\psi) \to g(\psi)) \land g(\psi) \to h(\Diamond\psi)))\} \to h(\varphi) \geq$$
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- Thus, now $\Gamma \vdash_{\mathcal{L}+N_\Box} \bigwedge\limits_{\psi \in \Sigma} \tau(\psi) \to \varphi$. By $N_\Box$ we get
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Truth Lemma

Witness lemma

$Rhg \leq g(\varphi)$ for all $g \in W$ implies $h(\Box\varphi) = 1$.

Fix $\tau(\psi) = (\overline{h(\Box\psi)} \to \psi) \wedge (\psi \to \overline{h(\Diamond\psi)})$.

- $\Rightarrow$ for each $c \in A$,
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- Using the axioms of $\mathcal{L}$, is easy to prove that
  $h(\bigwedge\limits_{\psi \in \Sigma} \Box\tau(\psi)) = 1$, and thus $h(\Box\varphi) = 1$ too.

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Witness Lemma suffices to prove $h(\Box\varphi) \geq \bigwedge_{g \in W} \{Rhg \to g(\varphi)\}$.

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Altogether prove completeness of $\mathcal{L} + N_\Box$ with respect to $\Vdash^{g}_{M_{\mathbf{A}(c)}}$.

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Answer to Q3 - constants-

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- Thus $\Gamma \nvdash_{\mathcal{Q}+N_{\Box}} \varphi \implies \Gamma \nvdash_{\mathcal{Q}+\mathcal{L}^c+N_{\Box}} \varphi \iff \Gamma \nVdash_{M_{\mathbf{A}(c)}} \varphi$. It is immediate to check that, for Γ, φ without constants, $\Gamma \Vdash_{M_{\mathbf{A}(c)}} \varphi \iff \Gamma \Vdash_{M_{\mathbf{A}}} \varphi$.

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- Axiomatizations without constant symbols are not clear out of very particular case studies (Ł, Gödel).
- Infinitarity of the semantical consequence relation seems to arise in the modal axiomatizations (even if there exists AS for the finitary companion at the propositional level)...

thank you!