Modal logics over finite residuated lattices

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- Modal expansions of lattice-based logics are in phase of development and understanding.
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  - only □ operator -with the usual lattice-valued interpretation
    Q1. Both □ and ◊ (! ◊x ≠ ¬□¬x)
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  - local deduction, global over crisp frames
    Q2. (general) Global deduction
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- Modal expansions of lattice-based logics are in phase of development and understanding.
- (Bou et. al., 2011) does a general study of axiomatizations of these logics over finite residuated lattices. Propose several open problems. We will address some of them:
  - only □ operator -with the usual lattice-valued interpretation
    Q1. Both □ and ◇ (! ◇x ≠ ¬□¬x)
  - local deduction, global over crisp frames
    Q2. (general) Global deduction
  - Q3. Is an axiomatization for the Global modal logic an axiomatization for the local one + \( \frac{x \rightarrow y}{\Box x \rightarrow \Box y} \)?
    (Q3'). Similar question restricting to crisp accessibility and adding \( \frac{x}{\Box x} \)
Preliminaries

- $A = \langle A, \cdot, \to, \land, \lor, 0, 1 \rangle$ is a (bounded, commutative, integral) residuated lattice when
  - $\langle A, \land, \lor, 1, 0 \rangle$ is a bounded lattice (with order denoted $\leq$),
  - $\langle A, \cdot, 1 \rangle$ is a commutative monoid and
  - for all $a, b, c \in A$ it holds $a \cdot b \leq c \iff a \leq b \to c$. 

- $A_{\text{c}} = \text{expansion of } A \text{ with constants } \{ a : a \in A \{ 1, 0 \} \}$.

- $F_{\text{m}} = \text{formula algebra built in the language of residuated lattices } [+ \text{ constants}]$.

- $\Gamma | \phi \iff \text{for any } h \in \text{Hom}(F_{\text{m}}, A)$, $h(\Gamma) \subseteq \{ 1 \}$ implies $h(\phi) = 1$. 

In the following $A$ will be finite.
\( \mathbf{A} = \langle A, \cdot, \rightarrow, \wedge, \vee, 0, 1 \rangle \) is a (bounded, commutative, integral) **residuated lattice** when

- \( \langle A, \wedge, \vee, 1, 0 \rangle \) is a bounded lattice (with order denoted \( \leq \)),
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- for all \( a, b, c \in A \) it holds \( a \cdot b \leq c \iff a \leq b \rightarrow c \).

\( \mathbf{A}^c = \) expansion of \( \mathbf{A} \) with constants \( \{ \overline{a} : a \in A \setminus \{1, 0\} \} \).
A = ⟨A, ⋅, →, ∧, ∨, 0, 1⟩ is a (bounded, commutative, integral) residuated lattice when

- ⟨A, ∧, ∨, 1, 0⟩ is a bounded lattice (with order denoted ≤),
- ⟨A, ⋅, 1⟩ is a commutative monoid and
- for all a, b, c ∈ A it holds a ⋅ b ≤ c ⇐⇒ a ≤ b → c.

A^c = expansion of A with constants \{\overline{a}: a ∈ A \setminus \{1, 0\}\}.

Fm = formula algebra built in the language of residuated lattices [+ constants].
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\begin{itemize}
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  \item \langle A, \cdot, 1 \rangle is a commutative monoid and
  \item for all \( a, b, c \in A \) it holds \( a \cdot b \leq c \iff a \leq b \rightarrow c \).
\end{itemize}

\( A^c \) = expansion of \( A \) with constants \( \{\bar{a} : a \in A \setminus \{1, 0\}\} \).

\( \text{Fm} = \) formula algebra built in the language of residuated lattices [+ constants].

\( \Gamma \models_A \varphi \) iff for any \( h \in \text{Hom}(\text{Fm}, A) \),

\[ h([\Gamma]) \subseteq \{1\} \text{ implies } h(\varphi) = 1. \]
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In the following \( A \) will be finite
\( \mathcal{M} = \langle W, R, e \rangle \) is a **A-Kripke model** when \( W \) is a non-empty set, \( R: W \times W \rightarrow A \) and \( e: W \times \mathcal{V} \rightarrow A \), extended uniquely in order to be in \( \text{Hom}(\text{Fm}, A) \) and
\[
e(v, \Box \varphi) = \bigwedge_{w \in W} \{ Rvw \rightarrow e(w, \varphi) \} \quad e(v, \Diamond \varphi) = \bigvee_{w \in W} \{ Rvw \cdot e(w, \varphi) \}
\]
It is said **crisp** if \( R \subseteq W \times W \).
\[ M = \langle W, R, e \rangle \] is a \textbf{A-Kripke model} when \( W \) is a non-empty set, \( R : W \times W \to A \) and \( e : W \times \mathcal{V} \to A \), extended uniquely in order to be in \( \text{Hom} (\text{Fm}, A) \) and

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It is said \textbf{crisp} if \( R \subseteq W \times W \).

\[ \Gamma \vDash_{M_A} \varphi \text{ iff for any A-Kripke model } M, \text{ and any } v \in W, \text{ if } e(v, [\Gamma]) \subseteq \{1\} \text{ then } e(v, \varphi) = 1. \]
\( \mathcal{M} = \langle W, R, e \rangle \) is a \textbf{A-Kripke model} when \( W \) is a non-empty set, \( R: W \times W \rightarrow A \) and \( e: W \times V \rightarrow A \), extended uniquely in order to be in \( \text{Hom}(\text{Fm}, A) \) and

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\( \Gamma \models_{M_A}^I \varphi \) iff for any \textbf{A-Kripke model} \( \mathcal{M} \), and any \( v \in W \), if \( e(v, [\Gamma]) \subseteq \{1\} \) then \( e(v, \varphi) = 1 \).

\( \Gamma \models_{M_A}^g \varphi \) iff for any \textbf{A-Kripke model} \( \mathcal{M} \), if for all \( v \in W \), it holds \( e(v, [\Gamma]) \subseteq \{1\} \) then for all \( v \in W \) it also holds \( e(v, \varphi) = 1 \).
\( \mathcal{M} = \langle W, R, e \rangle \) is an **A-Kripke model** when \( W \) is a non-empty set, \( R : W \times W \rightarrow A \) and \( e : W \times V \rightarrow A \), extended uniquely in order to be in \( \text{Hom}(Fm, A) \) and
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Same valid formulas.
Some comparisons with classical $K$

- No $K$. (Bou et. al) [$K$ is valid only if $Rvw$ is idempotent.]
(Some comparisons with classical \(K\))

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- No \(\Box = \neg \Diamond \neg\). \([\text{Only if } \neg \text{ is involutive (eg., MV algebras)}]\).
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- No $K$. (Bou et al) [$K$ is valid only if $Rvw$ is idempotent.]

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- Local classical modal logic enjoys $DT \implies$ usually we say "modal logic" for the set of valid formulas or the global consequence. No longer (necessarily) true - nor even LDT.
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$L_{\boxdot}^{A(c)} = \text{Axiomatization for } |=_{A(c)} +$

- $\Box 1,$
Existing axiomatization

For $A^c$ finite RL with canonical constants, Bou et. al propose an axiomatic system complete wrt. the no-$\diamond$ fragment of $\models_{MA(c)}$ (with constants).

$L_{\square}^{A(c)} = \text{Axiomatization for } \models_{A(c)} +$

- $\square 1,$
- $\square (\varphi \land \psi) \leftrightarrow (\square \varphi \land \square \psi),\]
For $A^c$ finite RL with canonical constants, Bou et. al propose an axiomatic system complete wrt. the no-$\Diamond$ fragment of $\vdash^l_{I_{A(c)}}$ (with constants).

$L_{\Box A(c)}^c = \text{Axiomatization for } \models_{A(c)} +$

- $\Box 1$,
- $\Box(\varphi \land \psi) \leftrightarrow (\Box \varphi \land \Box \psi)$,
- $\Box(\overline{c} \rightarrow \varphi) \leftrightarrow (\overline{c} \rightarrow \Box \varphi)$,
Existing axiomatization

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$L_{\Box}^{A(c)} = \text{Axiomatization for } \models_{A(c)} +$

- $\Box 1,$
- $\Box (\varphi \land \psi) \leftrightarrow (\Box \varphi \land \Box \psi),$
- $\Box (\lnot c \to \varphi) \leftrightarrow (\lnot c \to \Box \varphi),$  
- $\vdash \varphi \to \psi \text{ implies } \vdash \Box \varphi \to \Box \psi.$
Existing axiomatization

For $\mathbf{A}^c$ finite RL with canonical constants, Bou et. al propose an axiomatic system complete wrt. the no-$\Diamond$ fragment of $\mathcal{M}_{\mathbf{A}(c)}$ (with constants).

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- $\square 1,$
- $\square (\varphi \land \psi) \leftrightarrow (\square \varphi \land \square \psi),$
- $\square (\overline{c} \rightarrow \varphi) \leftrightarrow (\overline{c} \rightarrow \square \varphi),$
- $\vdash \varphi \rightarrow \psi$ implies $\vdash \square \varphi \rightarrow \square \psi.$

(For SI residuated lattices (with a unique coatom), adding $K$ and $\square (x \lor \overline{c}) \rightarrow (\square x \lor \overline{c})) \iff$ completeness wrt. the no-$\Diamond$ fragment of $\mathcal{M}_{\mathbf{C}_{\mathbf{A}}^c}.$)
Both modal operators

answer to Q1

For $\mathcal{L} = (\text{the previous A.S.}) + \Box(x \rightarrow \overline{c}) \rightarrow (\Diamond x \rightarrow \overline{c})$, $\mathcal{L}$ is complete with respect to $\models^I_{M_{A(c)}}$ (also concerning only idempotent/crisp frames completeness).
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**answer to Q1**

For $\mathcal{L} = (\text{the previous A.S.}) + \Box(x \to c) \to (\Diamond x \to c)$, $\mathcal{L}$ is complete with respect to $\vDash_{M_{A(c)}}^I$ (also concerning only idempotent/crisp frames completeness).

- Two RL equations concerning existing arbitrary infima/suprema are partially axiomatically represented

$$\bigwedge_{x \in X} x \to y = \bigvee X \to y$$

and

$$\bigwedge_{x \in X} y \to x = y \to \bigwedge X$$

Some small modifications in the canonical model defined by Bou. et. al. suffice to check completeness. [in defining Rvw] Same solution serves for the crisp case.
Both modal operators

Answer to Q1

For $\mathcal{L} = (\text{the previous A.S.}) + \Box(x \rightarrow \overline{c}) \rightarrow (\Diamond x \rightarrow \overline{c})$, $\mathcal{L}$ is complete with respect to $\Vdash_{M_{A(c)}}^L$ (also concerning only idempotent/crisp frames completeness).

- Two RL equations concerning existing arbitrary infima/suprema are partially axiomatically represented:
  \[
  \bigwedge_{x \in X} x \rightarrow y = \bigvee X \rightarrow y \quad \text{and} \quad \bigwedge_{x \in X} y \rightarrow x = y \rightarrow \bigwedge X
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**answer to Q1**

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For what concerns global logics...

It was not proven whether an axiomatic system for the global logic over $A/ A^{(c)}$ can be obtained by adding (the appropriated) necessity rule to an axiomatic system for the local one.
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**answer to Q2**

$L + N_\Box (= \frac{\varphi \rightarrow \psi}{\Box \varphi \rightarrow \Box \psi})$ is complete with respect to $\vdash_{M_{A(c)}}$ (also considering the no-$\Diamond$ restriction and the idempotent cases)
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It was not proven whether an axiomatic system for the global logic over $A/ A^{(c)}$ can be obtained by adding (the appropriated) necessity rule to an axiomatic system for the local one.

**answer to Q2**

$L + N_{\Box}(= \frac{\psi \rightarrow \varphi}{\Box \varphi \rightarrow \Box \psi})$ is complete with respect to $\vdash_{M_{A(c)}}^g$ (also considering the no-$\Diamond$ restriction and the idempotent cases)

- Clear that $\Gamma \vdash_{M_{A(c)}}^I \varphi \implies \Gamma \vdash_{M_{A(c)}}^g \varphi$. 
For what concerns global logics...

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- Clear that $\Gamma \vdash_{M_{A(c)}}^{l} \varphi \implies \Gamma \vdash_{M_{A(c)}}^{g} \varphi$.
- Thus soundness follows easily: $\Gamma \vdash_{\mathcal{L} + N_{\Box}} \varphi \implies \Gamma \vdash_{M_{A(c)}}^{g} \varphi$ (all axioms from $\mathcal{L}$ are sound in the global deduction, and so is $N_{\Box}$.)
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- Clear that $\Gamma \vdash_{M_{A(c)}}^l \varphi \iff \Gamma \vdash_{M_{A(c)}}^g \varphi$.
- Thus soundness follows easily: $\Gamma \vdash_{L + N_{\Box}} \varphi \iff \Gamma \vdash_{M_{A(c)}}^g \varphi$ (all axioms from $L$ are sound in the global deduction, and so is $N_{\Box}$.)

For the completeness direction, we will build appropriated canonical models.
Assume $\Gamma \not\models_{\mathcal{L}^+N\Box} \varphi$. We define the $\Gamma$-canonical model by:

1. $W = \{h \in Hom(Fm, A^{(c)}): h(C_{\mathcal{L}^+N\Box}(\Gamma)) = 1\}$,
Assume $\Gamma \not\models_{\mathcal{L}+\mathcal{N}_\Box} \varphi$. We define the $\Gamma$-canonical model by:

- $\mathcal{W} = \{ h \in \text{Hom}(\mathcal{Fm}, \mathcal{A}^{(c)}): h(\mathcal{C}_{\mathcal{L}+\mathcal{N}_\Box}(\Gamma)) = 1 \}$,
- $e(h, \varphi) = h(\varphi)$,
Assume $\Gamma \not\vDash_{\mathcal{L} + N \square} \varphi$. We define the $\Gamma$-canonical model by:

$W = \{ h \in \text{Hom}(Fm, A^{(c)}): h(C_{\mathcal{L} + N \square}(\Gamma)) = 1 \}$,

$e(h, \varphi) = h(\varphi)$,

$Rhg = \bigwedge_{\psi \in MFm} \{ ((h(\square \psi) \rightarrow g(\psi)) \land (g(\psi) \rightarrow h(\Diamond \psi))) \}$.
On the canonical model(s)

Assume $\Gamma \not\vdash_{\mathcal{L}+\mathcal{N}^\Box} \varphi$. We define the $\Gamma$-canonical model by:

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- $Rhg = \bigwedge_{\psi \in M\mathbf{Fm}} \{(h(\square \psi) \rightarrow g(\psi)) \land (g(\psi) \rightarrow h(\Diamond \psi))\}$,

(before proving the above is indeed an $\mathbf{A}^{(c)}$-Kripke model...)

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1. By definition of $W$ and $e$, the above is a global model for $\Gamma$, 

On the canonical model(s)

Assume $\Gamma \not\triangleright_{\mathcal{L}+\mathbb{N}_\Box} \varphi$. We define the $\Gamma$-canonical model by:

- $W = \{ h \in \text{Hom}(\mathcal{Fm}, A^{(c)}): h(C_{\mathcal{L}+\mathbb{N}_\Box}(\Gamma)) = 1 \}$,
- $e(h, \varphi) = h(\varphi)$,
- $Rh_g = \bigwedge_{\psi \in \text{MFm}} \{ (((h(\Box \psi) \to g(\psi)) \land (g(\psi) \to h(\Diamond \psi))) \}$,

(before proving the above is indeed an $A^{(c)}$-Kripke model...)

1. By definition of $W$ and $e$, the above is a global model for $\Gamma$,
2. $\Gamma \not\triangleright_{\mathcal{L}+\mathbb{N}_\Box} \varphi \implies C_{\mathcal{L}+\mathbb{N}_\Box}(\Gamma) \not\triangleright_{A^{(c)}} \varphi$, so there is $h \in W$ for which $h(\varphi) < 1$. 
On the canonical model(s)

Assume $\Gamma \not\vDash_{\mathcal{L}+N\Box} \varphi$. We define the $\Gamma$-canonical model by:

1. $W = \{ h \in \text{Hom}(\mathbf{Fm}, \mathbf{A}^{(c)}): h(C_{\mathcal{L}+N\Box}(\Gamma)) = 1 \}$, 
2. $e(h, \varphi) = h(\varphi)$, 
3. $Rhg = \bigwedge_{\psi \in \text{MFm}} \{ (((h(\Box\psi) \to g(\psi)) \land (g(\psi) \to h(\Diamond\psi))) \}$.

(before proving the above is indeed an $\mathbf{A}^{(c)}$-Kripke model...)

1. By definition of $W$ and $e$, the above is a global model for $\Gamma$,
2. $\Gamma \not\vDash_{\mathcal{L}+N\Box} \varphi \implies C_{\mathcal{L}+N\Box}(\Gamma) \not\vDash_{\mathbf{A}^{(c)}} \varphi$, so there is $h \in W$ for which $h(\varphi) < 1$.
3. So this model would indeed serve to prove $\Gamma \not\vDash_{M^{(c)}_A} \varphi$. 

Is the evaluation given a modal evaluation?
Truth Lemma

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- Prop. formulas are immediate, since the worlds are propositional homomorphisms.
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- Prop. formulas are immediate, since the worlds are propositional homomorphisms.

\[ h(\Box \varphi) \equiv \bigwedge_{g \in W} \{ Rhg \to g(\varphi) \} \]

- \( \leq \) direction is easy:

\[
Rh g \to g(\varphi) = \\
\bigwedge_{\psi \in MFm} \{ (((h(\Box \psi) \to g(\psi)) \land g(\psi) \to h(\Diamond \psi))) \} \to h(\varphi) \geq \\
(h(\Box \varphi) \to g(\varphi)) \to g(\varphi) \geq \\
h(\Box \varphi)
\]
Truth Lemma

Witness lemma

\(Rhg \leq g(\varphi)\) for all \(g \in W\) implies \(h(\Box \varphi) = 1.\)
Truth Lemma

Witness lemma

$Rhg \leq g(\varphi)$ for all $g \in W$ implies $h(\Box \varphi) = 1$.

Fix $\tau(\psi) = (\overline{h(\Box \psi)} \to \psi) \land (\psi \to \overline{h(\Diamond \psi)})$. 

Using the axioms of $L$, is easy to prove that $h(\overline{\bigwedge \psi \in \Sigma 2 \tau(\psi)}) = 1$, and thus $h(\overline{2 \varphi}) = 1$ too.
Truth Lemma

Witness lemma

\[ Rhg \leq g(\varphi) \text{ for all } g \in W \text{ implies } h(\Box \varphi) = 1. \]

Fix \( \tau(\psi) = (h(\Box \psi) \rightarrow \psi) \land (\psi \rightarrow h(\Diamond \psi)) \).

\[ \Rightarrow \text{ for each } c \in A, \]
\[ C_{L+\Box} \Gamma, \{ \overline{c} \rightarrow \tau(\psi) \}_{\psi \in MFM} \models A \overline{c} \rightarrow \varphi \quad (1) \]
Truth Lemma

Witness lemma

\[ Rhg \leq g(\varphi) \text{ for all } g \in W \text{ implies } h(\Box \varphi) = 1. \]

Fix \( \tau(\psi) = (h(\Box \psi) \to \psi) \land (\psi \to h(\Diamond \psi)) \).

\[ \Rightarrow \text{ for each } c \in A, \]
\[ \mathcal{C}_{L+\text{N}_\Box}(\Gamma), \{ \overline{c} \to \tau(\psi) \}_{\psi \in \text{MFm}} \models A \overline{c} \to \varphi \quad (1) \]

\[ \Rightarrow \text{ A finite, so for each } c \in A \text{ there is a finite } \Sigma_c \subset \text{MFm} \text{ for which } (1) \iff \mathcal{C}_{L+\text{N}_\Box}(\Gamma), \{ \overline{c} \to \tau(\psi) \}_{\psi \in \Sigma_c} \models A \overline{c} \to \varphi \]
Truth Lemma

Witness lemma

\[ Rhg \leq g(\varphi) \text{ for all } g \in W \text{ implies } h(\square \varphi) = 1. \]

Fix \( \tau(\psi) = (\overline{h(\square \psi)} \rightarrow \psi) \land (\psi \rightarrow \overline{h(\Diamond \psi)}) \).

- for each \( c \in A \),
  \[ C_{\mathcal{L}+N_\Box}(\Gamma), \{ \overline{c} \rightarrow \tau(\psi) \}_{\psi \in MFm} \models_A \overline{c} \rightarrow \varphi \quad (1) \]

- \( A \) finite, so for each \( c \in A \) there is a finite \( \Sigma_c \subset MFm \) for which \( (1) \iff C_{\mathcal{L}+N_\Box}(\Gamma'), \{ \overline{c} \rightarrow \tau(\psi) \}_{\psi \in \Sigma_c} \models_A \overline{c} \rightarrow \varphi \)

- Taking \( \Sigma = \bigcup_{c \in A} \Sigma_c \), we obtain \( C_{\mathcal{L}+N_\Box}(\Gamma) \models_A \bigwedge_{\psi \in \Sigma} \tau(\psi) \rightarrow \varphi \).
Truth Lemma

**Witness lemma**

\[ Rhg \leq g(\varphi) \text{ for all } g \in W \text{ implies } h(\square \varphi) = 1. \]

Fix \( \tau(\psi) = (\overline{h(\square \psi)} \rightarrow \psi) \land (\psi \rightarrow \overline{h(\lozenge \psi)}) \).

- \( \Rightarrow \) for each \( c \in A \),
  \[ C_{\mathcal{L}+\mathcal{N}\square}(\Gamma), \{ \overline{c} \rightarrow \tau(\psi) \} \psi \in MFm \modelsA \overline{c} \rightarrow \varphi \quad (1) \]
- \( A \) finite, so for each \( c \in A \) there is a finite \( \Sigma_c \subset MFm \) for which (1) \( \iff \) \( C_{\mathcal{L}+\mathcal{N}\square}(\Gamma), \{ \overline{c} \rightarrow \tau(\psi) \} \psi \in \Sigma_c \modelsA \overline{c} \rightarrow \varphi \)
- Taking \( \Sigma = \bigcup_{c \in A} \Sigma_c \), we obtain \( C_{\mathcal{L}+\mathcal{N}\square}(\Gamma) \modelsA \bigwedge_{\psi \in \Sigma} \tau(\psi) \rightarrow \varphi \).
- Thus, now \( \Gamma \vdash_{\mathcal{L}+\mathcal{N}\square} \bigwedge_{\psi \in \Sigma} \tau(\psi) \rightarrow \varphi \). By \( \mathcal{N}\square \) we get
  \[ \Gamma \vdash_{\mathcal{L}+\mathcal{N}\square} \square(\bigwedge_{\psi \in \Sigma} \tau(\psi)) \rightarrow \square \varphi. \]
Truth Lemma

Witness lemma

\[ Rhg \leq g(\varphi) \text{ for all } g \in W \text{ implies } h(\Box \varphi) = 1. \]

Fix \( \tau(\psi) = (h(\Box \psi) \rightarrow \psi) \land (\psi \rightarrow h(\Diamond \psi)) \).

\begin{itemize}
  \item \( \Rightarrow \) for each \( c \in A \),
    \[ C_{L + N_\Box} (\Gamma), \{ \overline{c} \rightarrow \tau(\psi) \}_{\psi \in MFm} \models A \overline{c} \rightarrow \varphi \quad (1) \]
  \item A finite, so for each \( c \in A \) there is a finite \( \Sigma_c \subset MFm \) for which (1) \( \iff \) \( C_{L + N_\Box} (\Gamma), \{ \overline{c} \rightarrow \tau(\psi) \}_{\psi \in \Sigma_c} \models A \overline{c} \rightarrow \varphi \)
  \item Taking \( \Sigma = \bigcup_{c \in A} \Sigma_c \), we obtain \( C_{L + N_\Box} (\Gamma) \models A \bigwedge_{\psi \in \Sigma} \tau(\psi) \rightarrow \varphi. \)
  \item Thus, now \( \Gamma \vdash_{L + N_\Box} \bigwedge_{\psi \in \Sigma} \tau(\psi) \rightarrow \varphi. \) By \( N_\Box \) we get
    \[ \Gamma \vdash_{L + N_\Box} \Box( \bigwedge_{\psi \in \Sigma} \tau(\psi)) \rightarrow \Box \varphi. \]
  \item Using the axioms of \( L \), is easy to prove that
    \[ h( \bigwedge_{\psi \in \Sigma} \Box \tau(\psi) ) = 1, \text{ and thus } h(\Box \varphi) = 1 \text{ too}. \]
\end{itemize}
Concluding the completeness

Witness Lemma suffices to prove \( h(\Box \varphi) \geq \bigwedge_{g \in W} \{ Rhg \rightarrow g(\varphi) \} \).
Concluding the completeness

Witness Lemma suffices to prove $h(\Box \varphi) \geq \bigwedge_{g \in W} \{Rh \rightarrow g(\varphi)\}$.

- If $c \leq Rh \rightarrow g(\varphi)$ for all $g \in W$, then $Rh \rightarrow g(\overline{c} \rightarrow \varphi) = 1$ for all $g \in W$. 

Altogether prove completeness of $L^+_{\mathcal{N}_2}$ with respect to $\models^g_{\mathcal{M}_\mathcal{A}(c)}$. 

Concluding the completeness

Witness Lemma suffices to prove $h(\Box \varphi) \geq \bigwedge_{g \in W} \{Rh_g \to g(\varphi)\}$.

- If $c \leq Rh_g \to g(\varphi)$ for all $g \in W$, then $Rh_g \to g(\overline{c} \to \varphi) = 1$ for all $g \in W$.
- The Lemma leads to $1 = h(\Box (\overline{c} \to \varphi)) = c \to h(\Box \varphi)$. 
Concluding the completeness

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\( h(\Diamond \varphi) = \bigvee_{g \in W} \{ Rhg \cdot g(\varphi) \} \) is proven similarly.

\[ \geq \] is now the easy one by definition.
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Witness Lemma suffices to prove \( h(\square \varphi) \geq \bigwedge_{g \in W} \{Rh g \rightarrow g(\varphi)\} \).

- If \( c \leq Rh g \rightarrow g(\varphi) \) for all \( g \in W \), then \( Rh g \rightarrow g(\overline{c} \rightarrow \varphi) = 1 \) for all \( g \in W \).
- The Lemma leads to \( 1 = h(\square(\overline{c} \rightarrow \varphi)) = c \rightarrow h(\square \varphi) \).

\( h(\Diamond \varphi) = \bigvee_{g \in W} \{Rh g \cdot g(\varphi)\} \) is proven similarly.

\( \geq \) is now the easy one by definition.

- If \( c \geq Rh g \cdot g(\varphi) \) for all \( g \in W \), then \( Rh g \rightarrow g(\varphi \rightarrow \overline{c}) = 1 \) for all \( g \in W \).
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Altogether prove completeness of $\mathcal{L} + N\Box$ with respect to $\models^g_{M_A(c)}$. 

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*Let a theory $T$ be \textit{fully-determined} if for each formula $\varphi$ there is a unique $c_\varphi \in A$ such that $\varphi \iff \overline{c_\varphi} \in T$. 

*Truth lemma as before (ingredients are essentially the same). 

\[ \Gamma \not\vdash Q + N_2 \varphi \iff C_Q + N_2(\Gamma) \not\models A(c_\varphi). \] 

Answer to Q3 - constants-

If $Q$ is an axiomatic system complete with respect to $\vdash M(\varphi)$, then $Q + N_2$ is complete with respect to $\vdash g M(c_\varphi)$.
*Let a theory $T$ be *fully-determined* if for each formula $\varphi$ there is a unique $c_\varphi \in A$ such that $\varphi \iff c_\varphi \in T$.

For $Q$ complete wrt. $\vdash^l_{M_{A(c)}}$, define the canonical model for $\Gamma$ by:

$$\mathcal{W} = \{ T : T \text{ is fully-determined maximally consistent } \models^{(c)}_A$$
$$\text{-theory and } C_{Q+N_{\Box}}(\Gamma) \subseteq T \}.$$
*Let a theory $T$ be fully-determined if for each formula $\varphi$ there is a unique $c_\varphi \in A$ such that $\varphi \leftrightarrow \overline{c_\varphi} \in T$. For $Q$ complete wrt. $\models_M^A(c)$, define the canonical model for $\Gamma$ by:

1. $\mathcal{W} = \{ T : T$ is fully-determined maximally consistent $\models_A^{(c)}$ -theory and $C_{Q+N\Box}(\Gamma) \subseteq T \}$,
2. $e(T, \varphi) = c_\varphi$,
*Let a theory $T$ be fully-determined if for each formula $\varphi$ there is a unique $c_\varphi \in A$ such that $\varphi \leftrightarrow \overline{c_\varphi} \in T$.

For $Q$ complete wrt. $\models^I_{M_A(c)}$, define the canonical model for $\Gamma$ by:

- $W = \{ T : T$ is fully-determined maximally consistent $\models_A^{(c)}$-theory and $C_{Q+N_{\square}}(\Gamma) \subseteq T \}$,
- $e(T, \varphi) = c_\varphi$,
- $RTS = \wedge_{\psi \in MFm} \{ \((e(T, \square \psi) \rightarrow e(S, \psi)) \land (e(S, \psi) \rightarrow e(T, \Diamond \psi))\) \}$.
General approach

*Let a theory $T$ be fully-determined if for each formula $\varphi$ there is a unique $c_\varphi \in A$ such that $\varphi \leftrightarrow \overline{c_\varphi} \in T$.

For $Q$ complete wrt. $\vdash^l_{M_A(c)}$, define the canonical model for $\Gamma$ by:

$\mathcal{W} = \{ T : T$ is fully-determined maximally consistent $\models^{(c)}_A$ -theory and $C_{Q+N^\Box}(\Gamma) \subseteq T \}$,

$e(T, \varphi) = c_\varphi$,

$RTS = \bigwedge_{\psi \in MFm} \{ ((e(T, \Box \psi) \rightarrow e(S, \psi)) \land (e(S, \psi) \rightarrow e(T, \Diamond \psi))) \}$,

*Truth lemma as before (ingredients are essentially the same).
General approach

*Let a theory \( T \) be \textit{fully-determined} if for each formula \( \varphi \) there is a unique \( c_{\varphi} \in A \) such that \( \varphi \leftrightarrow c_{\varphi} \in T \).

For \( Q \) complete wrt. \( \models_{M_{A(c)}}^{\bot} \), define the canonical model for \( \Gamma \) by:

\[
\mathcal{W} = \{ T : T \text{ is fully-determined maximally consistent } \models_{A}^{(c)} \text{-theory and } C_{Q+N_{\Box}}(\Gamma) \subseteq T \},
\]

\[
\mathcal{e}(T, \varphi) = c_{\varphi},
\]

\[
\mathcal{RTS} = \bigwedge_{\psi \in MF_{m}} \{ (\mathcal{e}(T, \Box \psi) \rightarrow \mathcal{e}(S, \psi)) \land (\mathcal{e}(S, \psi) \rightarrow \mathcal{e}(T, \Diamond \psi)) \},
\]

*Truth lemma as before (ingredients are essentially the same).

*\( \Gamma \models_{Q+N_{\Box}} \varphi \) iff \( C_{Q+N_{\Box}}(\Gamma) \models_{A(c)} \varphi \).
General approach

*Let a theory $T$ be fully-determined if for each formula $\varphi$ there is a unique $c_\varphi \in A$ such that $\varphi \leftrightarrow \overline{c_\varphi} \in T$.

For $Q$ complete wrt. $\models_{A(c)}$, define the canonical model for $\Gamma$ by:

1. $\mathcal{W} = \{ T : T$ is fully-determined maximally consistent $\models_{A}^{(c)}$ -theory and $C_{Q+N\Box}(\Gamma) \subseteq T \}$,
2. $e(T, \varphi) = c_\varphi$,
3. $RTS = \bigwedge_{\psi \in MFm} \{((e(T, \Box \psi) \rightarrow e(S, \psi)) \land (e(S, \psi) \rightarrow e(T, \Diamond \psi)))\}$,

*Truth lemma as before (ingredients are essentially the same).

If $Q$ is an axiomatic system complete with respect to $\models_{A(c)}$, then $Q + N\Box$ is complete with respect to $\models_{A(c)}$. 

Answer to Q3 - constants-

If $Q$ is an axiomatic system complete with respect to $\models_{A(c)}$, then $Q + N\Box$ is complete with respect to $\models_{A(c)}$. 

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*Truth lemma as before (ingredients are essentially the same).

If $Q$ is an axiomatic system complete with respect to $\models_{A(c)}$, then $Q + N\Box$ is complete with respect to $\models_{A(c)}$. 

Answer to Q3 - constants-

If $Q$ is an axiomatic system complete with respect to $\models_{A(c)}$, then $Q + N\Box$ is complete with respect to $\models_{A(c)}$. 

Without constant symbols

Answer to Q3

If $Q$ is an axiomatic system complete with respect to $\vdash^I_{MA}$, then $Q + N\Box$ is complete with respect to $\vdash^g_{MA}$.

Recall soundness was general.
Without constant symbols

Answer to Q3

If $Q$ is an axiomatic system complete with respect to $\vdash^l_{M_A}$, then $Q + N_\square$ is complete with respect to $\vdash^g_{M_A}$.

Recall soundness was general.

- Let $\mathcal{L}^c$ be the axioms including constants from $\mathcal{L}$. Then $Q + \mathcal{L}^c$ is complete with respect to $\vdash^g_{M_A(c)}$. 
Without constant symbols

**Answer to Q3**

If \( Q \) is an axiomatic system complete with respect to \( \vdash^I_M \), then \( Q + N_{\square} \) is complete with respect to \( \vdash^g_M \).

Recall soundness was general.

- Let \( \mathcal{L}^c \) be the axioms including constants from \( \mathcal{L} \). Then \( Q + \mathcal{L}^c \) is complete with respect to \( \vdash^I_{M^c} \).

- By induction on the derivation, if \( \Gamma, \varphi \) don’t have constants then \( \Gamma \vdash_{Q + \mathcal{L}^c + N_{\square}} \varphi \) implies \( \Gamma \vdash_{Q + N_{\square}} \varphi \).
Without constant symbols

Answer to Q3

If $Q$ is an axiomatic system complete with respect to $\models^I_M$, then $Q + N_{\square}$ is complete with respect to $\models^g_M$.

Recall soundness was general.

- Let $L^c$ be the axioms including constants from $L$. Then $Q + L^c$ is complete with respect to $\models^M_{A(c)}$.
- By induction on the derivation, if $\Gamma, \varphi$ don’t have constants then $\Gamma \vdash_{Q+L^c+N_{\square}} \varphi$ implies $\Gamma \vdash_{Q+N_{\square}} \varphi$.
- Thus $\Gamma \not\vdash_{Q+N_{\square}} \varphi \iff \Gamma \not\vdash_{Q+L^c+N_{\square}} \varphi \iff \Gamma \not\models_{M_{A(c)}} \varphi$. It is immediate to check that, for $\Gamma, \varphi$ without constants, $\Gamma \models_{M_{A(c)}} \varphi \iff \Gamma \models_{M_A} \varphi$. 
- Q3 without limitation to finite algebras seems likely to hold. However, the current proofs cannot surpass the lack of DT.
Very related questions

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- Axiomatizations without constant symbols are not clear out of very particular case studies (Ł, Gödel).
Very related questions

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- Axiomatizations without constant symbols are not clear out of very particular case studies (Ł, Gödel).
- Infinitarity of the semantical consequence relation seems to arise in the modal axiomatizations (even if there exists AS for the finitary companion at the propositional level)...