Topology from enrichment: the curious case of partial metrics

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(reporting on a joint paper with Dirk Hofmann)

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SUR QUELQUES POINTS DU CALCUL FONCTIONNEL;

Par M. **Maurice Fréchet** (Paris) *).

Adunanza del 22 aprile 1906.

*) Thèse présentée à la Faculté des Sciences de Paris pour obtenir le grade de Docteur ès Sciences.

49. *Introduction de l'écart.* — Lorsque nous appliquerons les résultats généraux de la Première Partie à des exemples concrets, nous reconnaîtrons d'abord que, dans chaque cas, on peut faire correspondre à tout couple d'éléments A, B un nombre $(A, B) \geqq 0$, que nous appellerons *l'écart des deux éléments* et qui jouit des deux propriétés suivantes: *a*) L'écart (A, B) n'est nul que si A et B sont identiques. *b*) Si A, B, C, sont trois éléments quelconques, on a toujours $(A, B) \leqq (A, C) + (C, B)$.

⋮

discerner si deux d'entre eux sont ou non identiques et tels, de plus, qu'à deux quelconques d'entre eux A, B, on puisse faire correspondre un nombre $(A, B) = (B, A) \geqq 0$

Metrics

Fréchet [1906]: Metric space (X, d) is a map $d\colon X \times X \to [0, +\infty]$ such that

- $0 \geq d(x, x)$ — zero self-distance
- $d(z, y) + d(y, x) \geq d(z, x)$ — triangular inequality
- $d(y, x) = d(x, y)$ — symmetry
- $d(x, y) = 0 = d(y, x) \Longrightarrow x = y$ — separatedness
- $d(x, y) \neq +\infty$ — finiteness

(X, d) "has balls", thus comes with its metric topology, whence the convergence criterion:

$$(x_n)_n \to x \text{ in } (X, d) \iff (d(x_n, x))_n \to 0 \text{ in } [0, +\infty].$$

(This is *hineininterpretierung* because General Topology did not exist in 1906—but Fréchet already had this convergence criterion.)

Examples—or not?!

Elements of $X = \{0,1\}^{\mathbb{N}}$ are **bitstreams**. Define

$$d(x,y) = \frac{1}{2^k}$$

where k is the place of the first letter in which $x = x^0x^1x^2...$ and $y = y^0y^1y^2...$ differ.

Convergence $(d(x_n, x))_n \to 0$ means that "the initial segments of the x_n resemble more and more to x"

Elements of $X = \{0,1\}^{\mathbb{N}} \cup \{0,1\}^*$ are **bitstreams and their finite approximations**. Slightly amend the previous definition:

$$d(x,y) = \frac{1}{2^k}$$

where k is the place of the first letter in which $x = x^0x^1x^2...$ and $y = y^0y^1y^2...$ differ *or no longer exist*.

Unfortunately d is not a metric: $d(x,x) \neq 0$ if $x \in \{0,1\}^*$.

In particular is $(d(x_n, x))_n \to 0$ an inadequate convergence criterion: $x, x, x, ... \not\to x$ whenever x is a finite word.

Partial metrics

From the late seventies, several computer scientists (with a strong mathematical background) were interested in the "bitstream" (or "data flow") example.
Main issue: **non-zero self-distance and "tighter" triangular inequality**.
It took more than a decade to settle upon the following axioms (Matthews [1994]):

A map $d\colon X \times X \to [0, +\infty]$ is ...

...a metric if:	... a partial metric if:
$0 \geq d(x,x)$	$d(y,x) \geq d(x,x) \vee d(y,y)$
$d(z,y) + d(y,x) \geq d(z,x)$	$d(z,y) - d(y,y) + d(y,x) \geq d(z,x)$
$d(y,x) = d(x,y)$	$d(y,x) = d(x,y)$
$d(x,y) = 0 = d(y,x) \implies x = y$	$d(x,y) = d(x,x) = d(y,y) = d(y,x) \implies x = y$
$d(x,y) \neq +\infty$	$d(x,y) \neq +\infty$

Convergence criterion for $(x_n)_n \to x$ ...

$(d(x_n, x))_n \to 0$	$(d(x_n, x))_n \to d(x,x)$

... is still inadequate! E.g. in the bitstream example, any constant sequence $x, x, x, \ldots$ converges to every initial word of x (as well as to x itself).

So we need more than a mere analogy here—we need a common generalisation.

Quantales, quantaloids

A (unital) **quantale** $Q = (Q, \bigvee, \circ, 1)$ is a monoid $(Q, \circ, 1)$ which is also a sup-lattice $(Q, \bigvee)$ such that all $f \circ -$ and $- \circ g$ preserve suprema. It is in particular residuated:

$$g \circ f \leq h \quad \iff \quad g \leq h \swarrow f \quad \iff \quad f \leq g \searrow h.$$

A **quantaloid** $\mathcal{Q}$ is a category with hom-sup-lattices $\mathcal{Q}(X, Y)$ such that all $f \circ -$ and $- \circ g$ preserve suprema; it is also residuated.

A quantale is thus exactly a quantaloid with a single object. Quantaloids often arise from universal constructions on quantales.

Diagonals

For f, g, d in any quantaloid $\mathcal{Q}$,

$$\exists d_0, d_1 : \quad \begin{tikzcd} A_0 \arrow[r, dotted, "d_0"] \arrow[d, "f"'] \arrow[dr, "d"] & B_0 \arrow[d, "g"] \\ A_1 \arrow[r, dotted, "d_1"'] & B_1 \end{tikzcd} \quad \Longleftrightarrow \quad \begin{tikzcd} A_0 \arrow[r, dotted, "g \searrow d"] \arrow[d, "f"'] \arrow[dr, "d"] & B_0 \arrow[d, "g"] \\ A_1 \arrow[r, dotted, "d \swarrow f"'] & B_1 \end{tikzcd}$$

In this case, say that $d\colon f \to g$ is a **diagonal** in $\mathcal{Q}$.

The quantaloid $\mathcal{D}(\mathcal{Q})$ of diagonals in $\mathcal{Q}$ is defined by the composition rule

$$\begin{tikzcd} \cdot \arrow[d, "f"'] \arrow[drr, "e \circ_g d"] & & \cdot \arrow[d, "h"] \\ \cdot & & \cdot \end{tikzcd} \quad = \quad \begin{tikzcd} \cdot \arrow[r, dotted] \arrow[d, "f"'] \arrow[dr, "d"] & \cdot \arrow[r, dotted] \arrow[d, "g"] \arrow[dr, "e"] & \cdot \arrow[d, "h"] \\ \cdot \arrow[r, dotted] & \cdot \arrow[r, dotted] & \cdot \end{tikzcd} \quad , \text{ e.g. } e \circ_g d = (e \swarrow g) \circ d$$

with identities $\begin{tikzcd} \cdot \arrow[d, "f"'] \arrow[dr, "f"] & \cdot \arrow[d, "f"] \\ \cdot & \cdot \end{tikzcd}$, and local order as in $\mathcal{Q}$.

Diagonals (2)

There is a full embedding

$$I\colon \mathcal{Q} \to \mathcal{D}(\mathcal{Q})\colon \left(A \xrightarrow{f} B \right) \mapsto \left(\begin{tikzcd} A \arrow[d, "1_A"'] \arrow[dr, "f"] & B \arrow[d, "1_B"] \\ A & B \end{tikzcd} \right)$$

and also a lax morphism

$$J_0\colon \mathcal{D}(\mathcal{Q}) \to \mathcal{Q}\colon \left(\begin{tikzcd} A_0 \arrow[d, "f"'] \arrow[dr, "d"] & B_0 \arrow[d, "g"] \\ A_1 & B_1 \end{tikzcd} \right) \mapsto \left(A_0 \xrightarrow{g \searrow d} B_0 \right).$$

(There is a J_1 too, but we will not need it.)

Note: even for a quantale Q, the diagonals form a quantaloid $\mathcal{D}(Q)$.

$[0, +\infty]$

Write $R = ([0, +\infty], \bigwedge, +, 0)$ for Lawvere's [1973] (commutative) quantale of positive real numbers (opposite of natural order!); residuation is $a \searrow b = b \swarrow a = 0 \vee (b - a)$. Diagonals are

$$x\colon a \to b \text{ in } \mathcal{D}(R) \iff \exists x_0, x_1 : \quad \begin{tikzcd} \cdot \arrow[r, dotted, "x_0"] \arrow[d, "a"'] \arrow[dr, "x"] & \cdot \arrow[d, "b"] \\ \cdot \arrow[r, dotted, "x_1"'] & \cdot \end{tikzcd} \iff x \geq a \vee b.$$

Composition of diagonals looks like:

$$\begin{tikzcd} \cdot \arrow[r, dotted, "x-b"] \arrow[d, "a"'] \arrow[dr, "x"] & \cdot \arrow[r, dotted, "y-c"] \arrow[d, "b"] \arrow[dr, "y"] & \cdot \arrow[d, "c"] \\ \cdot \arrow[r, dotted, "x-a"'] & \cdot \arrow[r, dotted, "y-b"'] & \cdot \end{tikzcd} \iff \begin{tikzcd} & b \arrow[dr, bend left, "y"] & \\ a \arrow[ur, bend left, "x"] \arrow[rr, bend right, "x-b+y"'] & & c \end{tikzcd}$$

Furthermore, R and $\mathcal{D}(R)$ compare via

$$\begin{tikzcd} R \arrow[r, shift left, "I"] & \mathcal{D}(R) \arrow[l, shift left, "J_0"] \end{tikzcd} \qquad \begin{array}{ccc} a & \mapsto & a\colon 0 \to 0 \\ x - b & \mapsfrom & x\colon a \to b \end{array}$$

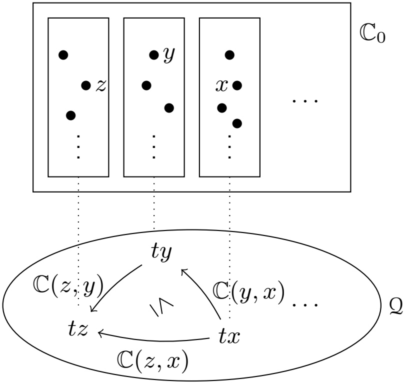
Enriched categories

Let $\mathcal{Q}$ be a (small) quantaloid. A $\mathcal{Q}$-**enriched category** $\mathbb{C}$ consists of:

- a set $\mathbb{C}_0$,
- a type function $t\colon \mathbb{C}_0 \to \mathsf{obj}(\mathcal{Q})$,
- a hom function $\mathbb{C}\colon \mathbb{C}_0 \times \mathbb{C}_0 \to \mathsf{arr}(\mathcal{Q})$

for which we have:

- $\mathbb{C}(y,x)\colon tx \to ty$,
- $1_{tx} \leq \mathbb{C}(x,x)$,
- $\mathbb{C}(z,y) \circ \mathbb{C}(y,x) \leq \mathbb{C}(z,x)$.



With the appropriate notion of $\mathcal{Q}$-**enriched functor**, we get a category $\mathsf{Cat}(\mathcal{Q})$, and with the appropriate notion of $\mathcal{Q}$-**enriched distributor**, we get a (large) quantaloid $\mathsf{Dist}(\mathcal{Q})$. The inclusion of $\mathsf{Cat}(\mathcal{Q})$ into $\mathsf{Dist}(\mathcal{Q})$ (mapping a functor on the left adjoint distributor that it represents) is the starting point for a very rich theory of $\mathcal{Q}$-enriched categories.

Enriched categories (2)

An **involution** on $\mathcal{Q}$ is a homomorphism

$$(-)^\circ \colon \mathcal{Q} \to \mathcal{Q}^{\mathsf{op}} \colon (\ A \xrightarrow{f} B\ ) \mapsto (\ A \xleftarrow{f^\circ} B\ )$$

which satisfies $f^{\circ\circ} = f$, and a $\mathcal{Q}$-category $\mathbb{C}$ is **symmetric** if $\mathbb{C}(y,x) = \mathbb{C}(x,y)^\circ$ holds.

Any $\mathcal{Q}$-category $\mathbb{C}$ can be symmetrised: define $\mathbb{C}_{\mathsf{s}}$ to have the same objects and types as $\mathbb{C}$, but with homs

$$\mathbb{C}_{\mathsf{s}}(y,x) = \mathbb{C}(y,x) \wedge \mathbb{C}(x,y)^\circ.$$

This defines the coreflector to the full embedding of symmetric $\mathcal{Q}$-categories into $\mathsf{Cat}(\mathcal{Q})$:

$$\begin{tikzcd} \mathsf{SCat}(\mathcal{Q}) \arrow[r, hook] \arrow[r, phantom, shift left=3, "\top"] & \mathsf{Cat}(\mathcal{Q}) \arrow[l, bend right, "(-)_{\mathsf{s}}"'] \end{tikzcd}$$

Metrics—again

Consider again the quantale (i.e. one-object quantaloid) $R = ([0, +\infty], \bigwedge, +, 0)$.

Following the previous definition, an R-category $\mathbb{C}$ consists of

- a set $\mathbb{C}_0$,
- a type function $t \colon \mathbb{C}_0 \to \mathsf{obj}(R)$,
- a hom function $\mathbb{C} \colon \mathbb{C}_0 \times \mathbb{C}_0 \to \mathsf{arr}(R)$,

such that, in R, we have

- $\mathbb{C}(y, x) \colon tx \to ty$,
- $1_{tx} \leq \mathbb{C}(x, x)$,
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such that, in R, we have

- $\mathbb{C}(y, x)\colon tx \to ty$,
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- $\cancel{\mathbb{C}(y, x)\colon tx \to ty}$,
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Metrics—again

Consider again the quantale (i.e. one-object quantaloid) $R = ([0, +\infty], \bigwedge, +, 0)$.

Following the previous definition, an R-category (X, d) consists of

- a set X,
- a function $d\colon X \times X \to R$,

such that

- $0 \geq d(x, x)$,
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This is a **generalised metric space** [Lawvere 1973] (and R-enriched functors are non-expansive maps.)

Considering the trivial involution on the commutative quantale R, a symmetric R-category is a symmetric generalised metric. Any generalised metric (X, d) can be symmetrised with the formula

$$d_{\mathsf{s}}(y, x) = d(y, x) \vee d(x, y).$$

(Also separatedness and finiteness can be dealt with categorically.)

Partial metrics—again

Again following the previous definitions, a $\mathcal{D}(R)$-category $\mathbb{C}$ consists of

- a set $\mathbb{C}_0$,
- a type function $t\colon \mathbb{C}_0 \to \mathsf{obj}(\mathcal{D}(R))$,
- a hom function $\mathbb{C}\colon \mathbb{C}_0 \times \mathbb{C}_0 \to \mathsf{arr}(\mathcal{D}(R))$,

such that, in $\mathcal{D}(R)$, we have

- $\mathbb{C}(y, x)\colon tx \to ty$,
- $1_{tx} \leq \mathbb{C}(x, x)$,
- $\mathbb{C}(z, y) \circ_{ty} \mathbb{C}(y, x) \leq \mathbb{C}(z, x)$.

Partial metrics—again

Again following the previous definitions, a $\mathcal{D}(R)$-category $\mathbb{C}$ consists of

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such that, in $\mathcal{D}(R)$, we have

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such that, in R, we have

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such that, in R, we have

- $\mathbb{C}(y, x)$ is a diagonal from tx to ty,
- $1_{tx} \leq \mathbb{C}(x, x)$,
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- $\mathbb{C}(y, x) \geq tx \vee ty$,
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Partial metrics—again

Again following the previous definitions, a $\mathcal{D}(R)$-category (X, p) consists of

- a set X,
- a function $p\colon X \times X \to R$,

such that

- $p(y, x) \geq p(x, x) \vee p(y, y)$,
- $p(z, y) - p(y, y) + p(y, x) \geq p(z, x)$.

Partial metrics—again

Again following the previous definitions, a $\mathcal{D}(R)$-category (X, p) consists of

- a set X,
- a function $p\colon X \times X \to R$,

such that

- $p(y, x) \geq p(x, x) \vee p(y, y)$,
- $p(z, y) - p(y, y) + p(y, x) \geq p(z, x)$.

Let us call this a **generalised partial metric space** [Höhle and Kubiak, 2011; Stubbe, 2014]. ($\mathcal{D}(R)$-enriched functors are non-expansive maps *that preserve self-distance*.)

(Also symmetry, separatedness and finiteness can be dealt with categorically.)

Metrics vs. partial metrics

The full embedding $I\colon R \to \mathcal{D}(R)\colon a \mapsto (a\colon 0 \to 0)$ determines the obvious full embedding

$$I\colon \mathsf{Cat}(R) \to \mathsf{Cat}(\mathcal{D}(R))$$

of metrics into partial metrics.

The lax morphism $J_0\colon \mathcal{D}(R) \to R\colon (x\colon a \to b) \mapsto x - b$ determines the functor

$$J_0\colon \mathsf{Cat}(\mathcal{D}(R)) \to \mathsf{Cat}(R)$$

which sends a partial metric space (X, p) to the metric space (X, p_0) in which

$$p_0(y, x) = p(y, x) - p(y, y).$$

(And remember that we can symmetrise.)

Topology from enrichment

Any subset $S \subseteq \mathbb{C}_0$ of a $\mathcal{Q}$-category $\mathbb{C}$ determines a full subcategory $\mathbb{S} \hookrightarrow \mathbb{C}$.

For $S \subseteq \mathbb{C}_0$, we want to compute the largest $S \subseteq \overline{S} \subseteq \mathbb{C}_0$ such that $i\colon \mathbb{S} \hookrightarrow \overline{\mathbb{S}}$ is **dense**:

for every $F, G\colon \overline{\mathbb{S}} \to \mathbb{X}$ in $\mathsf{Cat}(\mathcal{Q})$, if $F \circ i \cong G \circ i$ then $F \cong G$.

Doing distributor yoga (not explained here—sorry!), density of $i\colon \mathbb{S} \to \overline{\mathbb{S}}$ is equivalent to

$$i_* \otimes i^* = \mathrm{id}_{\overline{\mathbb{S}}} \quad \text{where} \quad \begin{tikzcd} \mathbb{S} \arrow[r, bend left, "i_*"] \arrow[r, phantom, "\perp"] & \overline{\mathbb{S}} \arrow[l, bend left, "i^*"] \end{tikzcd}.$$

(So density is "dual" to fully faithfulness.)

This, in turn, leads to the simple formula

$$\begin{aligned} \overline{S} &= \{x \in \mathbb{C}_0 \mid \mathbb{C}(x,x) = \bigvee_{s \in S} \mathbb{C}(x,s) \circ \mathbb{C}(s,x)\} \\ &= \{x \in \mathbb{C}_0 \mid 1_{tx} \leq \bigvee_{s \in S} \mathbb{C}(x,s) \circ \mathbb{C}(s,x)\}. \end{aligned}$$

Topology from enrichment (2)

So, for a subset $S \subseteq \mathbb{C}_0$ of a $\mathcal{Q}$-category $\mathbb{C}$, we put

$$\overline{S} = \{x \in \mathbb{C}_0 \mid 1_{tx} \leq \bigvee_{s \in S} \mathbb{C}(x,s) \circ \mathbb{C}(s,x)\}.$$

From this formula, it is straightforward to prove that:

- $S \mapsto \overline{S}$ is a closure operator on $\mathbb{C}_0$—call it the **categorical closure** on $\mathbb{C}$,
- enriched functors are continuous functions for the categorical closures,
- this makes for a functor $\mathsf{Cat}(\mathcal{Q}) \to \mathsf{Clos}$.

Furthermore,

- if each identity in $\mathcal{Q}$ is $\vee$-*irreducible*,

  for all X, $1_X \neq 0_X$ and if $1_X = f \vee g$ then $1_X = f$ or $1_X = g$,

  then the categorical closure on each $\mathbb{C}$ is topological (for integral $\mathcal{Q}$ this is an "iff"),
- an involutive $\mathcal{Q}$ is *strongly Cauchy-bilateral*,

  $$1_X \leq \bigvee_i g_i \circ f_i \Longrightarrow 1_X \leq \bigvee_i (f_i^\circ \wedge g_i) \circ (f_i \wedge g_i^\circ)$$

  iff the categorical closure on each $\mathbb{C}$ is identical to that on its symmetrisation $\mathbb{C}_s$.

Categorical topology on a metric space

In the quantale $R = ([0, +\infty], \bigwedge, +, 0)$, the unit 0 is $\wedge$-irreducible; so the categorical closure on any generalised metric space (X, d) is topological. For $S \subseteq X$, the general formula

$$x \in \overline{S} \iff 1_{tx} \leq \bigvee_{s \in S} \mathbb{C}(x, s) \circ \mathbb{C}(s, x)$$

becomes

$$x \in \overline{S} \iff 0 \geq \bigwedge_{s \in S} d(x, s) + d(s, x).$$

If the metric is symmetric, this is further equivalent to

$$\forall \varepsilon > 0 \; \exists s \in S : d(x, s) < \varepsilon.$$

In other words, the categorical topology is then the "usual" metric topology.

As the quantale R is strongly Cauchy-bilateral, the categorical topology on any generalised metric space (X, d) is identical to the metric topology on (X, d_{s}), the symmetrisation of (X, d).

Categorical topology on a partial metric space

The quantaloid $\mathcal{D}(R)$ has exactly one **zero object**:

$$\forall a \in \mathsf{obj}(\mathcal{D}(R)): \begin{tikzcd} \infty \arrow[r, bend left, "\exists!"] & a \arrow[l, bend left, "\exists!"] \end{tikzcd} \text{ in } \mathcal{D}(R).$$

Equivalently, this means that $1_\infty = 0_\infty$ in the quantale $\mathcal{D}(R)(\infty, \infty)$.

Therefore $\mathcal{D}(R)$ does not have $\wedge$-irreducible identities, and the categorical closure on a generalised partial metric space (X, p) **may not be topological**.

In fact, the general formula

$$\overline{S} = \{x \in \mathbb{C}_0 \mid 1_{tx} \leq \bigvee_{s \in S} \mathbb{C}(x, s) \circ \mathbb{C}(s, x)\}$$

produces in particular

$$\overline{\emptyset} = \{x \in X \mid p(x, x) = \infty\}$$

which **may not be empty**—and this is to be avoided if we want to say something useful about converging sequences.

Categorical topology on a partial metric space (2)

However, for any (X, p) we always have the categorical sum

$$X = \{x \in X \mid p(x,x) \neq \infty\} + \{x \in X \mid p(x,x) = \infty\}$$

(with obvious induced partial metrics) in $\mathsf{Cat}(\mathcal{D}(R))$, and the "interesting topology" happens in the **finitely typed summand** of (X, p).

Better still, if we write $\mathcal{D}(R)_{\mathsf{ft}}$ for the quantaloid $\mathcal{D}(R)$ without its zero object ∞, then

$$\mathsf{Cat}(\mathcal{D}(R)_{\mathsf{ft}}) \hookrightarrow \mathsf{Cat}(\mathcal{D}(R))$$

is the full coreflective subcategory of **finitely typed generalised partial metric spaces**; the coreflector sends a partial metric space (X, p) to its finitely typed summand.

The quantaloid $\mathcal{D}(\mathcal{Q})_{\mathsf{ft}}$ has all its identities $\wedge$-irreducible, so we can extract the categorical topology on the finitely typed summand of any partial metric space via

$$\begin{tikzcd}
\mathsf{Cat}(\mathcal{D}(R)_{\mathsf{ft}}) \arrow[r, hook, ""{name=A}] \arrow[d] & \mathsf{Cat}(\mathcal{D}(R)) \arrow[l, bend right, dotted, ""{name=B}] \arrow[d] \\
\mathsf{Top} \arrow[r, hook] & \mathsf{Clos}
\arrow[from=B, to=A, phantom, "\top"]
\end{tikzcd}$$

Categorical topology on a partial metric space (3)

So let now (X, p) be a finitely typed (summand of a) partial metric.

For $S \subseteq X$, the general formula

$$x \in \overline{S} \iff 1_{tx} \leq \bigvee_{s \in S} \mathbb{C}(x, s) \circ \mathbb{C}(s, x)$$

produces in particular

$$x \in \overline{S} \iff p(x, x) \geq \bigwedge_{s \in S} p(x, s) - p(s, s) + p(s, x).$$

By a happy coincidence (and because $p(x, x) \neq \infty$) this is further equivalent to

$$0 \geq \bigwedge_{s \in S} p_0(x, s) + p_0(s, x),$$

where (X, p_0) is the generalised metric obtained from (X, p) via $J_0 \colon \mathcal{D}(R) \to R$.

Furthermore symmetrising p_0, the categorical topology on a finitely typed partial metric space (X, p) is exactly the metric topology on $(X, (p_0)_{\mathsf{s}})$, where

$$(p_0)_{\mathsf{s}}(y, x) = p_0(y, x) \vee p_0(x, y) = (p(y, x) - p(y, y)) \vee (p(x, y) - p(x, x)).$$

Categorical topology on a partial metric space (4)

A diagrammatic summary of the situation:

$$
\begin{tikzcd}
\mathsf{Cat}(\mathcal{D}(R)) \arrow[rr, dotted, shift left] \arrow[ddr] & & \mathsf{Cat}(\mathcal{D}(R)_{\mathsf{ft}}) \arrow[ll, shift left] \arrow[ll, phantom, "\top"] \arrow[rr, shift left, "J_0"] \arrow[ddr] & & \mathsf{Cat}(R) \arrow[ll, shift left] \arrow[dr, shift left, "(-)_{\mathsf{s}}"] \arrow[ddl] & \\
& & & & & \mathsf{SCat}(R) \arrow[ul, shift left] \arrow[dll] \\
& \mathsf{Clos} & & \mathsf{Top} \arrow[ll] & &
\end{tikzcd}
$$

It is perhaps disappointing that there are no more "partially metrizable topologies" than there are metrizable topologies. Still, that doesn't mean that it is uninteresting to express topological phenomena in a partial metric space (X, p) directly in terms of the given partial metric p.

Convergence in a partial metric space

In a finitely typed (summand of a) generalised partial metric space (X, p), knowing now that its categorical topology is metrizable by $(p_0)_{\mathsf{s}}$, we have

$$
\begin{aligned}
(x_n)_n \to x \quad &\iff \quad ((p_0)_{\mathsf{s}}(x_n, x))_n \to 0 \\
&\iff \quad ((p(x_n, x) - p(x_n, x_n)) \vee (p(x, x_n) - p(x, x)))_n \to 0 \\
&\iff \quad (p(x_n, x) - p(x_n, x_n))_n \to 0 \leftarrow (p(x, x_n) - p(x, x))_n
\end{aligned}
$$

An easy computation shows that this is further equivalent to the "triple convergence"

$$
\begin{tikzcd}
 & (p(x_n, x_n))_n \arrow[d] & \\
(p(x_n, x))_n \arrow[r] & p(x, x) & (p(x, x_n))_n \arrow[l]
\end{tikzcd}
$$

Note in particular how the self-distances of the terms of a sequence $(x_n)_n$ must converge to the self-distance of the limit point!

This rules out the silly example in the partial metric space of bitstreams and words, of the constant sequence $x, x, x, \ldots$ "converging" to every initial word of x: it can now only converge to x itself.

More results

In our preprint on the arXiv, we have an extended discussion of ...

... 'divisible quantaloids',

... the (categorical/sequential) Cauchy completion of (X, p),

... the Hausdorff metric on (X, p),

... the exponentiability of (X, p).

Especially for Cauchy completion, our categorical method allows us to correct/complete some issues with earlier results in the literature.

Closing remarks [Fréchet, 1906]

Il fallait d'abord voir comment transformer les énoncés des théorèmes pour qu'ils conservent un sens dans le cas général. Il fallait ensuite, soit transcrire les démonstrations dans un langage plus général, soit, lorsque cela n'était pas possible, donner des démonstrations nouvelles et plus générales. Il s'est trouvé que les démonstrations que nous avons ainsi obtenues sont souvent aussi simples, et quelquefois même plus simples, que les démonstrations particulières qu'elles remplaçaient. Cela tient sans doute à ce que la position de la question obligeait à ne faire usage que de ses particularités vraiment essentielles.

"It turns out that our [more general] proofs are often as simple, and sometimes even simpler, than the particular proofs that they replace. This is because the way the question is put, forced us to use only its really essential features."