First-order interpolation from propositional interpolation: a proof theoretic approach on a semantic basis

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connect propositional and first-order interpolation

general methodology

$$\left.\begin{array}{r} \text{existence of suitable skolemizations } + \\ \text{existence of Herbrand expansions } + \\ \text{propositional interpolance} \end{array}\right\} \rightarrow \begin{array}{c} \text{first-order} \\ \text{interpolation.} \end{array}$$

This methodology is realized for lattice-based finitely-valued logics and can be extended to (fragments of) infinitely-valued logics.

the procedure

1. Develop a validity equivalent skolemization replacing all strong quantifiers in the valid formula $A \supset B$ to obtain the valid formula $A_1 \supset B_1$.
2. Construct a valid Herbrand expansion $A_2 \supset B_2$ for $A_1 \supset B_1$. Occurrences of $\exists x B(x)$ and $\forall x A(x)$ are replaced by suitable finite disjunctions $\bigvee B(t_i)$ and conjunctions $\bigwedge B(t_i)$.
3. Interpolate the propositionally valid formula $A_2 \supset B_2$ with the propositional interpolant I^*:

$$A_2 \supset I^* \quad \text{and} \quad I^* \supset B_2$$

are propositionally valid.

the procedure

4 Reintroduce weak quantifiers in $A_2 \supset I^*$ and $I^* \supset B_2$ to obtain valid formulas

$$A_1 \supset I^* \quad \text{and} \quad I^* \supset B_1.$$

5 Eliminate all function symbols and constants not in the common language of A_1 and B_1 by introducing suitable quantifiers in I^*. Let I be the result.

6 I is an interpolant for $A_1 \supset B_1$. $A_1 \supset I$ and $I \supset B_1$ are skolemizations of $A \supset I$ and $I \supset B$. Therefore I is an interpolant of $A \supset B$.

lattice-based finitely-valued logics

finite lattices $L = \langle W, \leq, \cup, \cap, 0, 1 \rangle$ where $\cup$, $\cap$, 0, 1 are *supremum*, *infimum*, *minimal element* and *maximal element*, $0 \neq 1$

A propositional language for L, $\mathcal{L}^0(L, V)$, $V \subseteq W$ is based on propositional variables, truth constants C_v for $v \in V$, $\vee$, $\wedge$, $\supset$.

A first-order language for L, $\mathcal{L}^1(L, V)$, $V \subseteq W$ is based on the usual first-order variables, predicates, truth constants C_v for $v \in V$, $\vee, \wedge, \supset, \exists, \forall$.

$\rightarrow$: $W \times W \Rightarrow W$ for $L = \langle W, \leq, \cup, \cap, 0, 1 \rangle$ is an admissible implication iff

$$u \rightarrow v = 1 \quad \Leftrightarrow \quad u \leq v,$$

$$u \leq v, f \leq g \quad \Rightarrow \quad v \rightarrow f \leq u \rightarrow g$$

skolemization

$$\left.\begin{array}{r}\text{existence of suitable skolemizations} + \\ \text{existence of Herbrand expansions} + \\ \text{propositional interpolance}\end{array}\right\} \rightarrow \begin{array}{c}\text{first-order} \\ \text{interpolation.}\end{array}$$

task: develop a validity equivalent skolemization replacing all strong quantifiers in the valid formula $A \supset B$ to obtain a valid formula $sk(A) \supset sk(B)$, s.t. the original formula can be reconstructed

skolemization

$A(sk(B))$ is defined as follows: replace strong quantifiers in B

$$\exists x C(x) \longrightarrow \bigvee_{i=1}^{|W|} C(f_i(\overline{x})), \qquad \forall x C(x) \longrightarrow \bigwedge_{i=1}^{|W|} C(f_i(\overline{x}))$$

where f_i are new function symbols and $\overline{x}$ are the weakly quantified variables of the scope
Skolem axioms are closed sentences

$$\forall \overline{x}(\exists y A(y, \overline{x}) \supset \bigvee_{i=1}^{|W|} A(f_i(\overline{x}), \overline{x}), \qquad \forall \overline{x}(\bigwedge_{i=1}^{|W|} A(f_i(\overline{x}), \overline{x}) \supset \forall y A(y, \overline{x}))$$

where f_i are new function symbols (Skolem functions)

skolemization

Lemma

1. $\models^1 A(B) \quad \Rightarrow \quad \models^1 A(sk(B))$
2. $S_1 \ldots S_k \models^1 A(sk(B)) \quad \Rightarrow \quad S_1 \ldots S_k \models^1 A(B)$
   for suitable Skolem axioms $S_1 \ldots S_k$
3. $S_1 \ldots S_k \models^1 A \quad \Rightarrow \quad \models^1 A$
   where $S_1 \ldots S_k$ *are Skolem axioms and* A *does not contain Skolem functions*

Herbrand expansions

$$\left.\begin{array}{r} \text{existence of suitable skolemizations} + \\ \text{existence of Herbrand expansions} + \\ \text{propositional interpolance} \end{array}\right\} \to \begin{array}{c} \text{first-order} \\ \text{interpolation.} \end{array}$$

task: construct a valid Herbrand expansion $A_H \supset B_H$ for $sk(A) \supset sk(B)$

expansion

Let A contain only weak quantifiers. An expansion of A is a quantifier free closed formula where

$$\exists x B(x) \text{ in } A \longrightarrow \bigvee B(t_i), \qquad \forall x C(x) \text{ in } A \longrightarrow \bigwedge C(s_j)$$

for some t_i, s_j.

Herbrand expansions can be constructed

A Herbrand expansion is a valid expansion.

Proposition

Let A contain only weak quantifiers. Then

$$\models^1 A \quad \Leftrightarrow \quad \textit{there is a valid Herbrand disjunction } A_H.$$

$$\left.\begin{array}{r}\text{existence of suitable skolemizations} + \\ \text{existence of Herbrand expansions} + \\ \text{propositional interpolance}\end{array}\right\} \to \begin{array}{c}\text{first-order} \\ \text{interpolation.}\end{array}$$

Theorem

Interpolation holds for $\mathbf{L}^0(L, V, \to)$

$$\Updownarrow$$

Interpolation holds for $\mathbf{L}^1(L, V, \to)$.

the interpolation theorem

Proof. One direction is trivial. Let's consider the the other.
$\Downarrow$: Assume $A \supset B \in \mathcal{L}(L, V)$ and $\models A \supset B$.

then, $\models sk(A) \supset sk(B)$

construct a Herbrand expansion $A_H \supset B_H$ from $sk(A) \supset sk(B)$

construct prop. interpolant to obtain $\models A_H \supset I^*, \quad \models I^* \supset B_H$

use $\models A(t) \supset \exists x A(x)$ and $\models \forall x A(x) \supset A(t)$

to obtain $\models sk(A) \supset I^*$ and $\models I^* \supset sk(B)$

order all terms $f(t)$ in I^* by inclusion (f is not in the common language) let $f^*(\bar{t})$ be the maximal such term in

$$\models sk(A) \supset I^* \quad \text{and} \quad \models I^* \supset sk(B)$$

- f^* is not in $sk(A)$:

  replace $f^*(\bar{t})$ by a fresh variable x : $\models sk(A) \supset I^*\{f^*(\bar{t}) \leftarrow x\}$

  but then, $\models sk(A) \supset \forall x I^*\{f^*(\bar{t}) \leftarrow x\}$

  by $\models \forall x I^*\{f^*(\bar{t}) \leftarrow x\} \supset I^*$ also $\models \forall x I^*\{f^*(\bar{t}) \leftarrow x\} \supset sk(B)$

  so we obtain

  $\models sk(A) \supset \forall x I^*\{f^*(\bar{t}) \leftarrow x\}$ and $\models \forall x I^*\{f^*(\bar{t}) \leftarrow x\} \supset sk(B)$

- $f^*(\bar{t})$ is not in $sk(B)$

  replace $f^*(\bar{t})$ by a fresh variable x : $\models I^*\{f^*(\bar{t}) \leftarrow x\} \supset sk(B)$

  but then, $\models \exists x I^*\{f^*(\bar{t}) \leftarrow x\} \supset sk(B)$

  by $\models I^* \supset \exists x I^*\{f^*(\bar{t}) \leftarrow x\}$ also $\models sk(A) \supset \exists x I^*\{f^*(\bar{t}) \leftarrow x\}$

  so we obtain

  $\models sk(A) \supset \exists x I^*\{f^*(\bar{t}) \leftarrow x\}$ and $\models \exists x I^*\{f^*(\bar{t}) \leftarrow x\} \supset sk(B)$

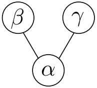
repeat until all functions and constants not in the common language are eliminated (among them the Skolem functions), let I be the result

I is an interpolant of $sk(A) \supset sk(B)$, therefore

$$A \supset I \quad \text{and} \quad I \supset B$$

example

The constant-domain intuitionistic Kripke frame $\mathcal{K}$



$\langle\{\alpha, \beta, \gamma\}, \leq\rangle$

is represented by the following lattice L

$$
\begin{tikzcd}[column sep=small]
 & \begin{pmatrix} 1 & & 1 \\ & 1 & \end{pmatrix} \arrow[d, dash, "\leq"] & \\
 & \begin{pmatrix} 1 & & 1 \\ & 0 & \end{pmatrix} \arrow[dl, dash, "\leq"'] \arrow[dr, dash, "\leq"] & \\
\begin{pmatrix} 0 & & 1 \\ & 0 & \end{pmatrix} \arrow[dr, dash, "\leq"'] & & \begin{pmatrix} 1 & & 0 \\ & 0 & \end{pmatrix} \arrow[dl, dash, "\leq"] \\
 & \begin{pmatrix} 0 & & 0 \\ & 0 & \end{pmatrix} &
\end{tikzcd}
$$

example

construct interpolant for $\exists x(B(x) \wedge \forall y C(y)) \supset \exists x(A(x) \vee B(x))$

example

construct interpolant for $\exists x(B(x) \land \forall y C(y)) \supset \exists x(A(x) \lor B(x))$

1. skolemization

$$\bigvee_{i=1}^{5}(B(c_i) \land \forall y C(y)) \supset \exists x(A(x) \lor B(x))$$

example

construct interpolant for $\exists x(B(x) \wedge \forall y C(y)) \supset \exists x(A(x) \vee B(x))$

1. skolemization

$$\bigvee_{i=1}^{5}(B(c_i) \wedge \forall y C(y)) \supset \exists x(A(x) \vee B(x))$$

2. Herbrand expansion

$$\bigvee_{i=1}^{5}(B(c_i) \wedge C(c_1)) \supset \bigvee_{i=1}^{5}(A(c_i) \vee B(c_i))$$

example

construct interpolant for $\exists x(B(x) \wedge \forall y C(y)) \supset \exists x(A(x) \vee B(x))$

1. skolemization
$$\bigvee_{i=1}^{5}(B(c_i) \wedge \forall y C(y)) \supset \exists x(A(x) \vee B(x))$$
2. Herbrand expansion
$$\bigvee_{i=1}^{5}(B(c_i) \wedge C(c_1)) \supset \bigvee_{i=1}^{5}(A(c_i) \vee B(c_i))$$
3. propositional interpolant
$$\bigvee_{i=1}^{5}(B(c_i) \wedge C(c_1)) \supset \bigvee_{i=1}^{5} B(c_i), \qquad \bigvee_{i=1}^{5} B(c_i) \supset \bigvee_{i=1}^{5}(A(c_i) \vee B(c_i))$$

example

3. propositional interpolant

$$\bigvee_{i=1}^{5}(B(c_i)\wedge C(c_1)) \supset \bigvee_{i=1}^{5} B(c_i), \qquad \bigvee_{i=1}^{5} B(c_i) \supset \bigvee_{i=1}^{5}(A(c_i)\vee B(c_i))$$

4. back to the Skolem form

$$\bigvee_{i=1}^{5}(B(c_i)\wedge\forall y C(y)) \supset \bigvee_{i=1}^{5} B(c_i), \qquad \bigvee_{i=1}^{5} B(c_i) \supset \exists x(A(x)\vee B(x))$$

example

3. propositional interpolant

$$\bigvee_{i=1}^{5}(B(c_i)\wedge C(c_1)) \supset \bigvee_{i=1}^{5} B(c_i), \qquad \bigvee_{i=1}^{5} B(c_i) \supset \bigvee_{i=1}^{5}(A(c_i)\vee B(c_i))$$

4. back to the Skolem form

$$\bigvee_{i=1}^{5}(B(c_i)\wedge\forall y C(y)) \supset \bigvee_{i=1}^{5} B(c_i), \qquad \bigvee_{i=1}^{5} B(c_i) \supset \exists x(A(x)\vee B(x))$$

5. eliminate function symbols and constants not in the common language

$$\bigvee_{i=1}^{5}(B(c_i)\wedge\forall y C(y)) \supset \exists z_1 \dots \exists z_5 \bigvee B(z_i),$$

$$\exists z_1 \dots \exists z_5 \bigvee B(z_i) \supset \exists x(A(x)\vee B(x))$$

example

5.
$$\bigvee_{i=1}^{5}(B(c_i) \wedge \forall y C(y)) \supset \exists z_1 \dots \exists z_5 \bigvee B(z_i),$$

$$\exists z_1 \dots \exists z_5 \bigvee B(z_i) \supset \exists x(A(x) \vee B(x))$$

6. use Skolem axiom

$$\exists x(B(x) \wedge \forall y C(y)) \supset \bigvee_{i=1}^{5} B(c_i) \wedge \forall y C(y)$$

to obtain original formula, Skolem axiom can be eliminated

Corollary

$$\textit{Interpolation holds for } \mathbf{L}^0(L, V, \to),$$

$$\models A \supset B, \quad A \supset B \textit{ contain only weak quantifiers}$$

$$\Downarrow$$

$$\textit{there is a quantifier-free interpolant for } A \supset B.$$

Corollary

Interpolation holds for $\mathbf{L}^0(L, V, \to)$*,*

$$\models A \supset B, \quad A \supset B \text{ contain only weak quantifiers}$$

$$\Downarrow$$

there is a quantifier-free interpolant for $A \supset B$*.*

Proposition

Let $L = \langle W, \leq, \cup, \cap, 0, 1 \rangle$*.*

i. $\mathbf{L}^0(L, \emptyset, \to)$ *(and therefore* $\mathbf{L}^1(L, \emptyset, \to)$*) never has the interpolation property.*

ii. $\mathbf{L}^0(L, W, \to)$ *(and therefore* $\mathbf{L}^1(L, \emptyset, \to)$*) always has the interpolation property.*

It is therefore reasonable to consider the function

$$\mathrm{SPEC}(L, \to) = \{V \mid \mathbf{L}^1(L, V, \to) \text{ interpolates}\}.$$

extensions to infinitely-valued logics

use described methodology to prove interpolation for (fragments of) infinitely-valued logics

- Gödel logic $G_{[0,1]}$, the logic of all linearly ordered Kripke frames with constant domains

its connectives can be interpreted as functions over the real interval $[0, 1]$

- $\bot$: logical constant for 0
- $\lor, \land, \exists, \forall$ are defined as *maximum*, *minimum*, *supremum*, *infimum*
- $\neg A$: $A \to \bot$, where

$$u \to v = \begin{cases} 1 & u \leq v \\ v & \text{else} \end{cases}$$

fragments of $G_{[0,1]}$

weak quantifier fragment of $G_{[0,1]}$

- admits Herbrand expansions (cut-free proofs in hypersequent calculi),
- as propositional Gödel logic interpolates, the weak quantifier fragment interpolates, too

fragment $A \supset B$, A, B prenex

- skolemization as in classical logic
- construct Herbrand expansion
- interpolate
- go back to Skolem form
- use immediate analogy of the 2nd ε-theorem to obtain the original formula