Extended Contact Logic

Philippe Balbiani[1] and Tatyana Ivanova[2]

[1]Institut de recherche en informatique de Toulouse
CNRS — Toulouse University
[2]Institute of Mathematics and Informatics
Bulgarian Academy of Sciences

Topology, Algebra and Categories in Logic 2017,
Prague, June 26-30

Contact Logic

We consider quantifier-free first-order language $\mathcal{L}(0, 1;\ +, \cdot, *;\ \leq, C)$.

Topological semantics of Contact Logic

- A **topological model** is a pair (X, V) where X is a topological space and $V : \ p \mapsto \ V(p) \in RC(X)$
- Truth conditions
  - $V(0) = \emptyset$, $V(\alpha^\star) = Cl(X \setminus V(\alpha))$, $V(\alpha + \beta) = V(\alpha) \cup V(\beta)$
  - $(X, V) \models C(\alpha, \beta)$ iff $V(\alpha) \cap V(\beta) \neq \emptyset$
  - $(X, V) \models \alpha \leq \beta$ iff $V(\alpha) \subseteq V(\beta)$

Contact Logic

Algebraic semantics of Contact Logic

- An **algebraic model** is a pair (B, V) where $(B, \leq_B, 0_B, 1_B, -_B, +_B, \cdot_B, C_B)$ is a contact algebra and $V: p \mapsto V(p) \in B$
- Truth conditions
  - $V(0) = 0_B$, $V(\alpha^\star) = -_B V(\alpha)$, $V(\alpha + \beta) = V(\alpha) +_B V(\beta)$
  - $(X, V) \models C(\alpha, \beta)$ iff $C_B(V(\alpha), V(\beta))$
  - $(X, V) \models \alpha \leq \beta$ iff $V(\alpha) \leq_B V(\beta)$

Contact Logic

Relational semantics of Contact Logic

- A **relational model** is a triple (W, R, V) where W is a nonempty set, R is a binary relation on W and $V : p \mapsto V(p) \subseteq W$
- Truth conditions
  - $V(0) = \emptyset$, $V(\alpha^\star) = W \setminus V(\alpha)$, $V(\alpha + \beta) = V(\alpha) \cup V(\beta)$
  - $(W, R, V) \models C(\alpha, \beta)$ iff $\exists s, t \in W\ (s \in V(\alpha)\ \&\ t \in V(\beta)\ \&\ sRt)$
  - $(W, R, V) \models \alpha \leq \beta$ iff $V(\alpha) \subseteq V(\beta)$

Contact Logic

Axiomatization (L_{min})

- equational theory of nondegenerate Boolean algebras
- $C(\alpha, \beta) \to \alpha \neq 0 \wedge \beta \neq 0$
- $C(\alpha, \beta) \wedge \alpha \leq \alpha' \wedge \beta \leq \beta' \to C(\alpha', \beta')$
- $C(\alpha + \beta, \gamma) \to C(\alpha, \gamma) \vee C(\beta, \gamma)$
- $C(\alpha, \beta + \gamma) \to C(\alpha, \beta) \vee C(\alpha, \gamma)$
- $\alpha \cdot \beta \neq 0 \to C(\alpha, \beta)$
- $C(\alpha, \beta) \to C(\beta, \alpha)$

Completeness

- φ is valid iff φ is derivable (Balbiani *et al.*, 2007)
- The set of all theorems of L_{min} is decidable (Balbiani *et al.*, 2007)

Complexity

- Satisfiability with respect to any determined class of relational models is in *NEXPTIME* (Balbiani *et al.*, 2007)
- Satisfiability with respect to the class of all topological models is *NP*-complete (Wolter and Zakharyaschev, 2000)
- Satisfiability with respect to the topological space $\mathcal{R}$ is *PSPACE*-complete (Wolter and Zakharyaschev, 2000)
- Satisfiability with respect to the class of all connected topological models is *PSPACE*-complete (Kontchakov *et al.*, 2013)

Internal connectedness

We consider the predicate c^o - internal connectedness. Let X be a topological space and $x \in RC(X)$. Let $c^o(x)$ means that $Int(x)$ is a connected topological space in the subspace topology. We prove that the predicate internal connectedness cannot be defined in the language of contact algebras. Because of this we add to the language a new ternary predicate symbol $\vdash$ which has the following sense: in the contact algebra of regular closed sets of some topological space $a, b \vdash c$ iff $a \cap b \subseteq c$.

Extended contact algebras

It turns out that the predicate c^o can be defined in the new language. We define *extended contact algebras* - Boolean algebras with added relations $\vdash$, C and c^o, satisfying some axioms, and prove that every extended contact algebra can be isomorphically embedded in the contact algebra of the regular closed subsets of some compact, semiregular, T_0 topological space with added relations $\vdash$ and c^o. So extended contact algebra can be considered an axiomatization of the theory, consisting of the formulas true in all topological contact algebras with added relations $\vdash$ and c^o.

Proposition 1

There does not exist a formula $A(x)$ in the language of contact algebras such that: for arbitrary topological space, for every regular closed subset x of this topological space, $c^o(x)$ iff $A(x)$ is valid in the algebra of regular closed subsets of the topological space.

Let X be a topological space. We define the relation $\vdash$ in $RC(X)$ in the following way: $a, b \vdash c$ iff $a \cap b \subseteq c$.

Proposition 2

Let X be a topological space. For every a in $RC(X)$, $c^o(a)$ iff $\forall b \forall c (b \neq 0 \wedge c \neq 0 \wedge a = b + c \rightarrow b, c \nvdash a^*)$.

Definition 0.1

Extended contact algebra (ECA, for short) is a system $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, \vdash, C, c^o)$, where $(B, \leq, 0, 1, \cdot, +, *)$ is a nondegenerate Boolean algebra, $\vdash$ is a ternary relation in B such that the following axioms are true:

(1) $a, b \vdash c \rightarrow b, a \vdash c$,

(2) $a \leq b \rightarrow a, a \vdash b$,

(3) $a, b \vdash a$,

(4) $a, b \vdash x,\ a, b \vdash y,\ x, y \vdash c \rightarrow a, b \vdash c$,

(5) $a, b \vdash c \rightarrow a \cdot b \leq c$,

(6) $a, b \vdash c \rightarrow a + x, b \vdash c + x$,

C is a binary relation in B such that for all a, $b \in B$: $aCb \leftrightarrow a, b \nvdash 0$. c^o is a unary predicate in B such that for all $a \in B$: $c^o(a) \leftrightarrow \forall b \forall c (b \neq 0 \wedge c \neq 0 \wedge a = b + c \rightarrow b, c \nvdash a^*)$.

Lemma 0.2

If $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, \vdash, C, c^o)$ is an ECA then C is a contact relation in B and hence (B, C) is a contact algebra.

The above lemma shows that the notion of ECA is a generalization of contact algebra.

The next lemma shows the standard topological example of ECA.

Lemma 0.3

Let X be a topological space and $RC(X)$ be the Boolean algebra of regular closed subsets of X. Let for $a, b, c \in RC(X)$:
aCb iff $a \cap b \neq \emptyset$,
$a, b \vdash c$ iff $a \cap b \subseteq c$
$c^0(a)$ iff $Int(a)$ is a connected subspace of X.
Then the Boolean algebra RC(X) with just defined relations is an ECA, called topological ECA over the space X.

Definition 0.4

Let $(B, \leq, 0, 1, \cdot, +, *, \vdash, C, c^o)$ be an ECA and $S \subseteq B$.

$S \vDash_0 x \overset{def}{\leftrightarrow} x \in S$

$S \vDash_{n+1} x \overset{def}{\leftrightarrow} \exists x_1, x_2 : \ x_1, x_2 \vdash x, \ S \vDash_{k_1} x_1, \ S \vDash_{k_2} x_2$, where $k_1, k_2 \leq n$

$S \vDash x \overset{def}{\leftrightarrow} \exists n : \ S \vDash_n x$

For proving a representation theorem of EC-algebras we need the following lemmas.

Lemma 0.5

If $S \models_n y$ *and* $S \subseteq S'$, *then* $S' \models_n y$.

Lemma 0.6

If $S \models_n y$ *and* $n \leq n'$, *then* $S \models_{n'} y$.

Lemma 0.7

If $S \models x$ *and* $x \leq y$, *then* $S \models y$.

Lemma 0.8

If $\{x\} \cup S \models y$, $\{y\} \cup S \models z$, *then* $\{x\} \cup S \models z$.

Lemma 0.9

If $\{x_1\} \cup S \models y$, $\{x_2\} \cup S \models y$, *then* $\{x_1 + x_2\} \cup S \models y$.

Lemma 0.10

Let $S \models x$. *Then there is a finite nonempty subset of* S S_0 *such that* $S_0 \models x$.

Lemma 0.11

Let $S = \{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_k\}$ *for some* $n, k > 0$ *and* $S \models x$. *Let* $a = a_1 \cdot \ldots \cdot a_n$, $b = b_1 \cdot \ldots \cdot b_k$. *Then* $a, b \vdash x$.

Abstract points

Definition 0.12

Let $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, \vdash, C, c^o)$ be an ECA. A subset of B Γ is an abstract point if the following conditions are satisfied:

1) $1 \in \Gamma$
2) $0 \notin \Gamma$
3) $a + b \in \Gamma \rightarrow a \in \Gamma$ or $b \in \Gamma$
4) $a, b \in \Gamma,\ a, b \vdash c \rightarrow c \in \Gamma$

Lemma 0.13

Let $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, \vdash, C, c^o)$ *be an ECA. Let* $A \neq \emptyset,\ A \subseteq B,\ a \in B,\ A \nvDash a$*. Then there exists an abstract point* Γ *such that* $A \subseteq \Gamma$ *and* $a \notin \Gamma$*.*

Theorem 0.14

(Representation theorem) Let $\underline{B} = (B, \leq, 0, 1, ., +, *, \vdash, C, c^o)$ *be an ECA. Then there is a compact, semiregular,* T_0 *topological space* X *and an embedding* h *of* $\underline{B}$ *into the topological ECA over* X*.*

X is the set of all abstract points of $\underline{B}$ and h is the well known Stone embedding mapping.

Extended contact logic

We consider a quantifier-free first-order language corresponding to ECA and a logic for ECA.

We consider the language $\mathcal{L}'$ with equality which has:
- constants: 0, 1
- functional symbols: $+$, $\cdot$, $*$
- predicate symbols: $\leq$, $\vdash$, c^o

Extended contact logic

We consider the logic L which has the following:

• axioms:

- the axioms of the classical propositional logic
- the axioms of Boolean algebra - as axiom schemes
- the axioms $(1), \ldots, (6)$ of ECA - as axiom schemes
- the axiom scheme:

(Ax c^o) $c^o(p) \wedge q \neq 0 \wedge r \neq 0 \wedge p = q + r \rightarrow q, r \nvdash p^*$

• rules:

- MP
- (Rule c^o) $\frac{\alpha \rightarrow (p \neq 0 \wedge q \neq 0 \wedge a = p + q \rightarrow p, q \nvdash a^*) \text{ for all variables } p,\ q}{\alpha \rightarrow c^o(a)}$, where α is a formula, a is a term.

We also consider the logic L_{Axc^o} which is obtained from L by removing the rule (Rule c^o).

Extended contact logic

Theorem 0.15 (Completeness theorem with respect to algebraic and topological semantics)

For every formula α in $\mathcal{L}'$ the following conditions are equivalent:
(i) α is a theorem of L;
(ii) α is true in all ECA;
(iii) α is true in all ECA over a compact, T_0, semiregular topological space.

Proposition 3

The rule (Rule c^o) is not admissible in L_{Axc^o}.

Proposition 4

L is decidable.

Relational semantics of extended contact logic

A relational frame is a structure (W, R), where W is a nonempty set and R is a ternary relation on subsets of W. We say that a relational frame is nice if the following conditions are satisfied for any subsets of W A, B, C, X and Y:

(1) $R(A, B, C) \rightarrow R(B, A, C)$,
(2) $A \subseteq B \rightarrow R(A, A, B)$,
(3) $R(A, B, A)$,
(4) $R(A, B, X),\ R(A, B, Y),\ R(X, Y, C) \rightarrow R(A, B, C)$,
(5) $R(A, B, C) \rightarrow A \cap B \subseteq C$,
(6) $R(A, B, C) \rightarrow R(A \cup X, B, C \cup X)$.

Relational semantics of extended contact logic

Let (W, R) be a nice relational frame. We consider the Boolean algebra of all subsets of W - $\underline{B} = (B, \subseteq, \emptyset, W, \cap, \cup, *)$, where $A^* = W - A$ for any subset of W A. We define in $\underline{B}$ the relations $\vdash$, C and c^o in the following way:
$A, B \vdash C$ iff $R(A, B, C)$,
ACB iff $A, B \nvdash \emptyset$,
$c^o(A)$ iff $\forall B \forall C(B \neq \emptyset \wedge C \neq \emptyset \wedge A = B \cup C \rightarrow B, C \nvdash A^*)$ for any subsets of W A, B and C.
Obviously $\underline{B}$ is an ECA. We call it the relational ECA over (W, R). We say that a formula is true in (W, R) if it is true in the ECA over (W, R).

Theorem 0.16 (Completeness theorem)

For every formula α the following conditions are equivalent:
(i) α is a theorem of L;
(ii) α is true in all finite nice relational frames with number of the elements less or equal to $2^{3 \cdot h_n}$, where $h_n = 2^{(2^n)}$ and n is the number of the variables of α.

Definition 0.17

Let W be a nonempty set. We define an ECA of kind 3 in the following way: this is a structure
$\underline{B} = (B, \emptyset, W, \subseteq, *, \cup, \cdot, \vdash, C, c^o)$, where $B = 2^W$,
$*$ is a unary function such that:
1) $a \cup a^* = W$,
2) $(a^*)^* = a$,
3) $W^* = \emptyset$,
4) $(a \cup b)^* \subseteq a^*$,
5) $c \subseteq a^* \cap b^* \rightarrow c \subseteq (a \cup b)^*$,
$a \cdot b \overset{def}{=} (a^* \cup b^*)^*$,
$a, b \vdash c \overset{def}{\leftrightarrow} a \cap b \subseteq c$,
$aCb \overset{def}{\leftrightarrow} a, b \nvdash \emptyset$,
$c^o(a) \overset{def}{\leftrightarrow} (\forall b, c \in B)(b \neq \emptyset,\ c \neq \emptyset,\ a = b \cup c \rightarrow b, c \nvdash a^*)$
for any a, b, $c \in B$.

Another semantics of extended contact logic

Proposition 5

Every ECA of kind 3 is an ECA.

Theorem 0.18

(Representation theorem) Every finite ECA is isomorphic to an ECA of kind 3.

Theorem 0.19 (Completeness theorem)

For every formula α the following conditions are equivalent:
(i) α is a theorem of L;
(ii) α is true in all finite ECA of kind 3 with number of the elements less or equal to $2^{(2^{3 \cdot h_n})}$, where $h_n = 2^{(2^n)}$ and n is the number of the variables of α.

Thank you very much!