Antistructural completeness in propositional logics

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Introduction

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The (anti)structural completion of a logic is the strongest logic with the same (anti)theorems. Interestingly, unlike the structural completion, the antistructural completion of a logic need not always exist.

Our main goal is to provide several equivalent characterizations of such completions under some mild conditions. In particular, antistructural completeness turns out to be closely connected to semisimplicity.

Preliminaries: logics

A logic is a relation between sets of formulas and formulas, denoted $\Gamma \vdash \varphi$, which satisfies some natural conditions:

$$\varphi \vdash_{\mathcal{L}} \varphi \qquad \text{(reflexivity)}$$

$$\Gamma \vdash_{\mathcal{L}} \varphi \Rightarrow \Gamma, \Delta \vdash_{\mathcal{L}} \varphi \qquad \text{(monotonicity)}$$

$$\Gamma \vdash_{\mathcal{L}} \varphi \Rightarrow \sigma[\Gamma] \vdash_{\mathcal{L}} \sigma\varphi \text{ for each substitution } \sigma \qquad \text{(structurality)}$$

$$\Gamma \vdash_{\mathcal{L}} \delta \text{ for each } \delta \in \Delta \text{ and } \Delta, \Pi \vdash_{\mathcal{L}} \varphi \Rightarrow \Gamma, \Pi \vdash_{\mathcal{L}} \varphi \qquad \text{(cut)}$$

A logic $\mathcal{L}$ is finitary if the following holds:

$$\Gamma \vdash_{\mathcal{L}} \varphi \Rightarrow \Gamma' \vdash_{\mathcal{L}} \varphi \text{ for some finite } \Gamma' \subseteq \Gamma \qquad \text{(finitarity)}$$

For finitary logics, cut is equivalent to the following condition:

$$\Gamma \vdash_{\mathcal{L}} \varphi \text{ and } \varphi, \Delta \vdash_{\mathcal{L}} \psi \Rightarrow \Gamma, \Delta \vdash_{\mathcal{L}} \psi \qquad \text{(finitary cut)}$$

Structural completions

A theorem of a logic $\mathcal{L}$ is a formula φ which is designated in every model of $\mathcal{L}$, i.e. φ such that $\emptyset \vdash_{\mathcal{L}} \varphi$. The set of all theorems of $\mathcal{L}$ is denoted $\mathrm{Thm}\,\mathcal{L}$.

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Given a logic $\mathcal{L}$, its axiomatic part $\mathsf{Ax}_{\mathcal{B}}\,\mathcal{L}$ is defined as:

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Equivalently, $\mathsf{Ax}_{\mathcal{B}}\,\mathcal{L}$ is the smallest extension of $\mathcal{B}$ with the same theorems as $\mathcal{L}$, i.e. the extension of $\mathcal{B}$ by the rules $\emptyset \vdash \varphi$ for $\varphi \in \mathsf{Thm}\,\mathcal{L}$.

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We have $\Gamma \vdash_{\sigma\mathcal{L}} \varphi$ if and only if the rule $\Gamma \vdash \varphi$ is admissible in $\mathcal{L}$, that is:

$\emptyset \vdash_{\mathcal{L}} \sigma\varphi$ whenever $\emptyset \vdash_{\mathcal{L}} \sigma[\Gamma]$ for each substitution σ.

Antitheorems: definition

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Some logics have antitheorems but no finite antitheorems, for example the principal Gödel logic with positive rational constants.

Antitheorems: basic properties

The following versions of monotonicity, structurality, and cut hold:

$$\Gamma \vdash_{\mathcal{L}} \emptyset \Rightarrow \Gamma \vdash_{\mathcal{L}} \varphi \qquad \text{(right monotonicity)}$$

$$\Gamma \vdash_{\mathcal{L}} \emptyset \Rightarrow \Gamma, \Delta \vdash_{\mathcal{L}} \emptyset \qquad \text{(left monotonicity)}$$

$$\Gamma \vdash_{\mathcal{L}} \emptyset \Rightarrow \sigma[\Gamma] \vdash_{\mathcal{L}} \emptyset \text{ for each substitution } \sigma \qquad \text{(structurality)}$$

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If $\mathcal{L}$ is finitary, then the following version of finitarity holds:

$$\Gamma \vdash_{\mathcal{L}} \emptyset \Rightarrow \Gamma' \vdash_{\mathcal{L}} \emptyset \text{ for some finite } \Gamma' \subseteq \Gamma \qquad \text{(finitarity)}$$

Structurality and finitarity may fail if we define antitheorems by $\Gamma \vdash \mathrm{Fm}$.

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Given a logic $\mathcal{L}$, its explosive part $\mathsf{Exp}_{\mathcal{B}}\,\mathcal{L}$ is defined as:

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$\mathrm{Exp}_{\mathcal{B}}$ is an interior operator. Moreover, $\mathrm{Exp}_{\mathcal{B}} \bigcap_{i \in I} \mathcal{L}_i = \bigcap_{i \in I} \mathrm{Exp}_{\mathcal{B}}\,\mathcal{L}_i$.

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An explosive extension of $\mathcal{B}$ is an extension $\mathcal{L}$ such that $\mathrm{Exp}_{\mathcal{B}}\,\mathcal{L} = \mathcal{L}$, i.e. an extension axiomatized by a set of rules of the form $\Gamma \vdash \emptyset$ relative to $\mathcal{B}$.

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The explosive extensions of $\mathcal{B}$ form a completely distributive sublattice of $\mathrm{Ext}\,\mathcal{B}$, denoted $\mathrm{Exp}\,\mathrm{Ext}\,\mathcal{B}$, such that $\bigvee_{i \in I} \mathcal{L}_i = \bigcup_{i \in I} \mathcal{L}_i$ for $\mathcal{L}_i \in \mathrm{Exp}\,\mathrm{Ext}\,\mathcal{B}$.

Explosive parts: examples

Let us look at the explosive parts of some known logics: Ł is Łukasiewicz logic, $\mathcal{BD}$ is the Dunn–Belnap logic, $\mathcal{LP}$ is the Logic of Paradox.

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$\mathsf{Exp}_{\mathcal{BD}}\,\mathcal{CL}$ is axiomatized relative to $\mathcal{BD}$ by the rules $\chi_n \vdash \emptyset$ for $n \geq 1$ with:

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$\mathsf{Exp}_{\text{Ł}}\,\mathcal{CL}$ is axiomatized by the following three rules:

$$p \to \neg p, \neg p \to p \vdash \emptyset$$
$$p \to \neg p, (p \cdot q) \to \neg(p \cdot q) \vdash \emptyset$$
$$\neg p \to p, \neg q \to q, (p \cdot q) \to \neg(p \cdot q) \vdash \emptyset$$

(These hold in an MV-algebra iff it has a homomorphism into $\{0, 1\}$.)

Explosive parts: digression

Explosive parts are useful when computing logics given by a product of matrices if we know the logics given by the factors.

Proposition

$\operatorname{Log} \prod_{i \in I} \boldsymbol{A}_i = \bigcap_{i \in I} \operatorname{Log} \boldsymbol{A}_i \cup \bigcup_{i \in I} \operatorname{Exp}_{\mathcal{B}} \operatorname{Log} \boldsymbol{A}_i$, *where the matrices* $\boldsymbol{A}_i$ *are non-trivial models of* $\mathcal{B}$.

Corollary

If $\mathcal{B} = \operatorname{Log} \boldsymbol{A}$ *and* $\mathcal{L} = \operatorname{Log} \boldsymbol{B}$, *then* $\operatorname{Exp}_{\mathcal{B}} \mathcal{L} = \operatorname{Log} \boldsymbol{A} \times \boldsymbol{B}$.

Antistructural completions

The antistructural completion of $\mathcal{L}$, denoted $\alpha\mathcal{L}$, is the strongest extension of $\mathcal{L}$ (provided that it exists) with the same antitheorems as $\mathcal{L}$.

A logic $\mathcal{L}$ is said to be antistructurally complete if $\alpha\mathcal{L} = \mathcal{L}$.

$\mathcal{L}'$ has the same antitheorems as $\mathcal{L} \Leftrightarrow \mathsf{Exp}_{\mathcal{B}}\,\mathcal{L} \subseteq \mathcal{L}' \subseteq \alpha\mathcal{L}$ (if $\alpha\mathcal{L}$ exists).

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Unlike structural completions, antistructural completions need not exist.

Example: consider the principal Gödel logic with rational constants c_q for $q \in \mathbb{Q} \cap [0, 1]$. Adding an arbitrary c_q for $q > 0$ as a theorem does not yield any new antitheorems. But adding all of them yields the trivial logic.

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We want a sufficient condition for existence and a useful description of $\alpha\mathcal{L}$.

Antiadmissible rules

A rule $\Gamma \vdash \varphi$ is called antiadmissible in $\mathcal{L}$ if:

$$\Delta, \sigma[\Gamma] \vdash_{\mathcal{L}} \emptyset \text{ whenever } \Delta, \sigma\varphi \vdash_{\mathcal{L}} \emptyset \text{ for each subst. } \sigma \text{ and each } \Delta$$

Proposition

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The antiadmissible rules of a logic satisfy reflexivity, monotonicity, structurality, and finitary cut (but not necessarily cut).

If a rule does not add new antitheorems, then it is antiadmissible. The converse does not necessarily hold in general.

Proposition

If $\mathcal{L}$ has a finite antitheorem and its antiadmissible rules are closed under cut, then $\alpha\mathcal{L}$ exists and consists precisely of the antiadmissible rules.

The maximal consistency property (MCP)

We say that a logic enjoys the maximal consistency property (MCP) if each consistent theory extends to a maximal consistent one. That is:

if $\Gamma \nvdash_{\mathcal{L}} \emptyset$, then there is a max. $\Delta \supseteq \Gamma$ such that $\Delta \nvdash_{\mathcal{L}} \emptyset$

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Observation

Each finitary logic with a (finite) antitheorem enjoys the MCP.

On the other hand, the principal Gödel logic with rational constants has a finite antitheorem but not the MCP.

Proposition

Let $\mathcal{L}$ be a logic which enjoys the MCP. Then $\Gamma \vdash \varphi$ is antiadmissible in $\mathcal{L}$ if and only if it is valid in $\langle \boldsymbol{Fm}, \Gamma \rangle$ for each max. consistent $\mathcal{L}$-theory Γ.

Simplicity

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If F is a maximal non-trivial $\mathcal{L}$-filter on $\boldsymbol{A}$, we call F and $\langle \boldsymbol{A}, F \rangle$ simple.

A logic $\mathcal{L}$ is semisimple if each theory is an intersection of simple theories.

It is τ-semisimple if each $\mathcal{L}$-filter is an intersection of simple $\mathcal{L}$-filters.

Main theorem

Theorem (Existence and characterization of antistr. completions)

If $\mathcal{L}$ has a finite antitheorem and enjoys the MCP (in particular, if $\mathcal{L}$ is finitary and has an antitheorem), then the following are equivalent:

(i) $\Gamma \vdash_{\alpha\mathcal{L}} \varphi$.

(ii) $\Gamma \vdash \varphi$ *is antiadmissible in* $\mathcal{L}$.

(iii) $\Gamma \vdash \varphi$ *is valid in all simple matrices over* $\boldsymbol{Fm}$.

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If $\mathcal{L}$ is moreover protoalgebraic, then these are equivalent to:

(iv) $\sigma\varphi, \Delta \vdash_{\mathcal{L}} \emptyset$ *implies* $\sigma\Gamma, \Delta \vdash_{\mathcal{L}} \emptyset$ *for each invertible substitution* σ.

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If $\mathcal{L}$ is moreover finitary, then these are equivalent to:

(vi) $\varphi, \Delta \vdash_{\mathcal{L}} \emptyset$ *implies* $\Gamma, \Delta \vdash_{\mathcal{L}} \emptyset$.

(vii) $\Gamma \vdash \varphi$ *holds in all simple models of* $\mathcal{L}$.

Connections to semisimplicity

In the well-behaved cases, $\alpha\mathcal{L}$ is the "semisimple part" of $\mathcal{L}$.

Theorem

Let $\mathcal{L}$ be a finitary protoalgebraic logic with an antitheorem. Then the theories of $\alpha\mathcal{L}$ are precisely the intersections of simple theories of $\mathcal{L}$. If $\mathcal{L}$ moreover has a countable language, then this holds for all filters of $\alpha\mathcal{L}$.

Corollary

If $\mathcal{L}$ is a finitary protoalgebraic logic with an antitheorem, then $\alpha\mathcal{L}$ is semisimple. If $\mathcal{L}$ also has a countable language, then $\alpha\mathcal{L}$ is τ-semisimple.

Examples

$\mathcal{BD}$ = the Belnap–Dunn logic
$\mathcal{LP}$ = the Logic of Paradox

$$\mathcal{ECQ} = \mathcal{BD} + p, -p \vdash q$$
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Example: $\alpha\mathcal{BD} = \mathcal{LP}$. $\alpha\mathcal{ECQ} = \mathcal{ETL}$.

Example: $\alpha\mathcal{IL} = \mathcal{CL}$.

Proof: the simple BL-algebras are precisely the simple MV-algebras.

Examples

$\mathcal{BD}$ = the Belnap–Dunn logic
$\mathcal{LP}$ = the Logic of Paradox

$\mathcal{ECQ} = \mathcal{BD} + p, -p \vdash q$
$\mathcal{ETL} = \mathcal{BD} + p, -p \vee q \vdash q$.

Example: $\alpha\mathcal{BD} = \mathcal{LP}$. $\alpha\mathcal{ECQ} = \mathcal{ETL}$.

Example: $\alpha\mathcal{IL} = \mathcal{CL}$.

Proof: the simple BL-algebras are precisely the simple MV-algebras.

Example: an axiomatic extension of FL_{ew} is antistructurally complete if and only if it validates the axiom $p \vee \neg p^n$ for some n.

Proof: a variety of FL_{ew}-algebras is semisimple if and only if it satisfies $x \vee \neg x^n = 1$ for some n. (Kowalski)

Teaser...

Actually, protoalgebraicity is an overkill here. A weaker property, which we call protonegationality, suffices:

$$\Omega\Gamma \subseteq \Omega\Delta \text{ if } \Gamma \text{ is an } \mathcal{L}\text{-theory and } \Delta \text{ is a simple } \mathcal{L}\text{-theory.}$$

Example: the $\{\wedge, \vee, \sim\}$ fragment of intuitionistic logic.

The theory of protonegational logics is (for logics with the MCP) nearly as smooth, although not as powerful, as the theory of protoalgebraic logics.

In particular, protonegational logics form the appropriate framework for the study of inconsistency lemmas initiated recently by Raftery.

Conclusion

Antistructural completeness is the natural dual of structural completeness, when we focus our attention on antitheorems rather than theorems.

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Thank you for your attention.