Canonical extensions of archimedean vector lattices with strong order unit

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Dedicated to Bjarni Jónsson (1920-2016)

Canonical extensions of Boolean algebras

Canonical extensions of Boolean algebras with operators were introduced in the seminal paper of **Jónsson** and **Tarski** (1951). They were generalized to distributive lattices with operators by **Gehrke** and **Jónsson** (1994), lattices with operators by **Gehrke** and **Harding** (2001), and further to posets (**Gehrke**, **Priestley** 2008, **Gehrke**, **Jansana**, **Palmigiano** 2013).

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Stone duality provides motivation for the definition of canonical extensions. The canonical extension B of a Boolean algebra A is isomorphic to the powerset $\wp(X)$ of the Stone space X of A, and the embedding $e : A \to B$ is realized as the inclusion of the Boolean algebra $\mathrm{Clop}(X)$ of clopen subsets of X into $\wp(X)$.

Definition. A **canonical extension** of a Boolean algebra A is a pair $A^\sigma = (B, e)$, where B is a complete Boolean algebra and $e : A \to B$ is a Boolean monomorphism satisfying:

1. (Density) Each $x \in B$ is a join of meets (and hence a meet of joins) of elements of $e[A]$.

2. (Compactness) For $S, T \subseteq A$, from

$$\bigwedge e[S] \leq \bigvee e[T]$$

it follows that

$$\bigwedge e[S'] \leq \bigvee e[T']$$

for some finite $S' \subseteq S$ and $T' \subseteq T$.

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2. An ℓ-group A is a **vector lattice** if A is an $\mathbb{R}$-vector space and for each $0 \leq a \in A$ and $0 \leq \lambda \in \mathbb{R}$, we have $\lambda a \geq 0$.

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3. A is **archimedean** if for each $a, b \in A$, whenever $na \leq b$ for each $n \in \mathbb{N}$, then $a \leq 0$.

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3. A is **archimedean** if for each $a, b \in A$, whenever $na \leq b$ for each $n \in \mathbb{N}$, then $a \leq 0$.
4. A has a **strong order unit** if there is $u \in A$ such that for each $a \in A$ there is $n \in \mathbb{N}$ with $a \leq nu$. When u exists we call A **bounded**.

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Objects in **bav** can be viewed as normed spaces in the usual way, where the uniform norm on A is given by

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Let $A \in$ **bav**. Then A is **uniformly complete** if it is complete with respect to the uniform norm.

Let A be an archimedean vector lattice with strong order unit.

By the Yosida representation, A is represented as a uniformly dense vector sublattice of the vector lattice $C(Y)$ of all continuous real-valued functions on the Yosida space Y of A.

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By the Yosida representation, A is represented as a uniformly dense vector sublattice of the vector lattice $C(Y)$ of all continuous real-valued functions on the Yosida space Y of A.

Moreover, if A is uniformly complete, then A is isomorphic to $C(Y)$.

Since Y is compact, every continuous real-valued function on Y is bounded. Therefore, $C(Y)$ is a vector sublattice of the vector lattice $B(Y)$ of all bounded real-valued functions on Y.

The inclusion $C(Y) \hookrightarrow B(Y)$ has many similarities with the inclusion $\mathrm{Clop}(X) \hookrightarrow \wp(X)$.

In particular, the inclusion $C(Y) \hookrightarrow B(Y)$ satisfies the density axiom. However, it never satisifes the compactness axiom.

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For example, if Y is a singleton, then both $C(Y)$ and $B(Y)$ are isomorphic to $\mathbb{R}$.

If S is the set of positive real numbers and T the set of negative real numbers, then $\bigwedge S \leq \bigvee T$ as both are 0, but there are not finite subsets $S' \subseteq S$ and $T' \subseteq T$ with $\bigwedge S' \leq \bigvee T'$.

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A vector lattice A is **Dedekind complete** if every subset of A bounded above has a least upper bound, and hence every subset of A bounded below has a greatest lower bound.

Let **dbav** be the full subcategory of **bav** consisting of Dedekind complete objects of **bav**.

Canonical extensions of vector lattices

Definition. A **canonical extension** of $A \in \mathbf{bav}$ is a pair $A^\sigma = (B, e)$, where $B \in \mathbf{dbav}$ and $e : A \to B$ is a unital vector lattice monomorphism satisfying:

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1. (Density) Each $x \in B$ is a join of meets of elements of $e[A]$.
2. (Compactness) For $S, T \subseteq A$ and $0 < \varepsilon \in \mathbb{R}$, from

$$\bigwedge e[S] + \varepsilon \leq \bigvee e[T]$$

it follows that

$$\bigwedge e[S'] \leq \bigvee e[T']$$

for some finite $S' \subseteq S$ and $T' \subseteq T$.

The compactness axiom

If $A = B = \mathbb{R}$, then we saw that the original compactness axiom does not hold.

Recall the example. If $S = (0, \infty)$ and $T = (-\infty, 0)$, then $\bigwedge S \leq \bigvee T$ but there are not finite subsets $S' \subseteq S$ and $T' \subseteq T$ with $\bigwedge S' \leq \bigvee T'$.

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If $S, T \subseteq A$ with $\bigwedge S + \varepsilon \leq \bigvee T$, then as $\mathbb{R}$ is totally ordered, there is $s \in S$ and $t \in T$ with $s \leq t$.

Thus, the inclusion $A \hookrightarrow B$ satisfies the new compactness axiom.

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Corollary. The inclusion $C^*(X) \hookrightarrow B(X)$ is a canonical extension of $C^*(X)$ iff X is compact.

The Yosida space

Let $A \in \mathbf{bav}$. An **ℓ-ideal** of A is a subgroup I of A satisfying

$$a \in I \text{ and } |b| \leq |a| \text{ imply } b \in I.$$

If M is a maximal ℓ-ideal of A, then $A/M \cong \mathbb{R}$.

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For $A \in \mathbf{bav}$, the **Yosida space** of A is the set $Y(A)$ of maximal ℓ-ideals of A equipped with the topology whose closed sets are the sets of the form

$$Z(I) := \{M \in Y(A) : I \subseteq M\}$$

where I is an ℓ-ideal of A.

Existence

It is well known that the Yosida space Y of A is compact Hausdorff, $e : A \to C(Y)$ is an embedding, and $e[A]$ separates points, where $e(a)$ is the continuous function defined by

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if $a + M = \lambda + M$.

Theorem. Suppose $A \in \mathbf{bav}$. Then $e : A \to B(Y)$ is a canonical extension of A.

Some observations

(1) It is well known that if A is a Boolean algebra and $A^\sigma = (B, e)$ is a canonical extension of A, then $e : A \to B$ is an isomorphism iff the Stone space of A is finite, which is equivalent to A being finite.

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While in **bav** we still have that $e : A \to A^\sigma$ is an isomorphism iff the Yosida space Y is finite, it is no longer true that this is equivalent to A being finite. It is well known that Y is finite iff $A \cong \mathbb{R}^n$ for some n.

Therefore, in **bav**, the vector lattices $\mathbb{R}^n$ play the role of finite Boolean algebras with respect to canonical extensions.

(2) It is easy to see that canonical extensions of Boolean algebras do not preserve any existing strictly infinite joins or meets.

For, suppose A is a Boolean algebra and $a = \bigvee T$ in A. If $e(a) = \bigvee e[T]$, then the compactness axiom yields a finite $T' \subseteq T$ with $e(a) = \bigvee e[T'] = e(\bigvee T')$. Thus, a is a finite join in A.

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On the other hand, since the compactness axiom for bounded archimedean vector lattices is different, the above fact is not true in **bav**.

Let $A = C([0,1])$. Then $B([0,1])$ is a canonical extension of $C([0,1])$, and 0 is a strictly infinite join of $(-\infty, 0)$ which is preserved by the embedding $C([0,1]) \hookrightarrow B([0,1])$.

(3) Let A be a Boolean algebra, X the Stone space of A, and X_{disc} the discrete topology on X. Then it is well known that if $A^\sigma = (B, e)$ is a canonical extension of A, then the Stone space of B is homeomorphic to $\beta(X_{\text{disc}})$.

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Similarly, let $A \in \mathbf{bav}$ and Y be the Yosida space of A. If $A^\sigma = (B, e)$ is a canonical extension of A, then the Yosida space of B is $\beta(Y_{\mathsf{disc}})$.

Thus, from the topological perspective, our definition of canonical extensions for **bav** provides a natural generalization of the definition of canonical extensions for Boolean algebras.

Uniqueness

Let (B, e) and (B', e') be two canonical extensions of A. Define $\alpha : B \to B'$ as follows. First, if $y \in B$ is **closed** (that is, a meet from $e[A]$), we set

$$\alpha(y) = \bigwedge\{e'(a) : a \in A \text{ and } y \leq e(a)\}.$$

Define α in general for each $x \in B$ by

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Theorem. The map α is an isomorphism in **bav**. Thus, any two canonical extensions of $A \in$ **bav** are isomorphic.

Functorality

The canonical extension of A is, up to isomorphism, $(B(Y), e)$, where Y is the Yosida space of A and $e : A \to B(Y)$ is the composition of the Yosida embedding $A \to C(Y)$ and the inclusion $C(Y) \to B(Y)$.

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Thus, on objects, A^σ is obtained by applying to A the composition

$$\begin{tikzcd} \mathbf{bav} \arrow[r, "Y"] \arrow[rrr, bend right, "(-)^\sigma"'] & \mathbf{KHaus} \arrow[r, "F"] & \mathbf{Set} \arrow[r, "B"] & \mathbf{dbav}, \end{tikzcd}$$

where **KHaus** is the category of compact Hausdorff spaces, **Set** is the category of sets, and $F : \mathbf{KHaus} \to \mathbf{Set}$ is the forgetful functor.

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For, as the map $A \to B(Y)$ is monic for each $A \in \mathbf{bav}$, if $(-)^\sigma$ were a reflector, then it would be a monoreflector. Therefore, $(-)^\sigma$ would be a bireflector. Thus, the map $A \to B(Y)$ would be epic.

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But then the image of A would be uniformly dense in $B(Y)$. However, by Yosida duality, the image of A is uniformly dense in $C(Y)$. Hence, if the image of A were uniformly dense in $B(Y)$, then $B(Y) = C(Y)$, which is false for Y infinite.

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A similar argument shows that the canonical extension functor is not a reflector in the Boolean algebra setting.

Intrinsic characterization of canonical extensions

Let $B \in \mathbf{dbav}$. Then B is uniformly complete, so by Yosida duality, B is isomorphic to $C(Y(B))$, and hence B is a commutative ring with 1.

Therefore, the idempotents $\mathrm{Id}(B)$ of B form a Boolean algebra. Since B is Dedekind complete, $\mathrm{Id}(B)$ is complete.

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Therefore, the idempotents $\mathrm{Id}(B)$ of B form a Boolean algebra. Since B is Dedekind complete, $\mathrm{Id}(B)$ is complete.

Since the canonical extension of a Boolean algebra A is isomorphic to the powerset of the Stone space of A, the underlying Boolean algebra of the canonical extension of A is always complete and atomic.

The same is true for the idempotents of the canonical extension of $A \in \mathbf{bav}$. Let Y be its Yosida space.

If B is the underlying vector lattice of the canonical extension of A, then B is isomorphic to the ring $B(Y)$.

Thus, the idempotents of B correspond to characteristic functions on Y, so $\mathrm{Id}(B)$ is isomorphic to the powerset of Y, which is a complete and atomic Boolean algebra.

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Theorem. The following are equivalent for $B \in \mathbf{dbav}$.

1. B is realized as the canonical extension of some $A \in \mathbf{bav}$.
2. $\mathrm{Id}(B)$ is a complete and atomic Boolean algebra.
3. B has essential socle.

We further justify our use of the term "canonical extension" to describe the extension $e : A \to A^\sigma$ by showing how canonical extension for **bav** is a lifting of canonical extension in the category **BA** of Boolean algebras with Boolean homomorphisms.

Let **Stone** be the category of Stone spaces with continuous maps, and let $X : \mathbf{BA} \to \mathbf{Stone}$ be the Stone duality functor.

$$\begin{tikzcd}
\mathbf{BA} \arrow[r, "(-)^\sigma"] \arrow[d, "X"'] & \mathbf{BA} \arrow[d, "X"] \\
\mathbf{Stone} \arrow[d, "C"'] & \mathbf{Stone} \arrow[d, "C"] \\
\mathbf{bav} \arrow[r, "(-)^\sigma"'] & \mathbf{bav}
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For **BA** we have $(-)^\sigma = \wp \circ X$, where $\wp$ is the powerset functor; and for **bav** we have $(-)^\sigma = B \circ Y$.

The diagram commutes up to isomorphism of functors.

Thus, the canonical extension functor in **bav** lifts that of **BA**.

Thanks for listening, and thanks to the organizers for their work setting up this conference.