# The Free Algebra in a Two-sorted Variety of Probability Algebras

TACL 2017

Tomáš Kroupa[1] Vincenzo Marra[2]

[1]Czech Academy of Sciences

[2]Università degli Studi di Milano

# Probability

**Standard probability theory**

Finitely-additive probability is a function

$$P\colon A \to [0,1]$$

where $A$ is a Boolean algebra, $P$ satisfies $P(\top) = 1$ and

$$\text{If } a \wedge b = \bot, \text{ then } P(a \vee b) = P(a) + P(b) \qquad \text{for all } a, b \in A.$$

- All probability functions $P$ are $\sigma$-additive in the Stone representation.
- The **domain** and the **co-domain** of $P$ are sets of different sorts:

  events / probability degrees

# Probability and logic

**Hájek-style probability logic** for reasoning about uncertainty:

- 2-level syntax for formulas $\varphi$ representing events and formulas $P\varphi$ speaking about probability of $\varphi$
- Łukasiewicz logic makes it possible to axiomatize probability and introduce calculus, which gives meaning to expressions such as

$$P(\varphi \vee \psi) \to (P\varphi \oplus P\psi) \qquad \text{or} \qquad \varphi \to \psi \vdash P\varphi \to P\psi$$

with unary modality $P$ evaluated in $[0,1]_{MV}$

# Towards algebraic semantics for Hájek's probability logic

Algebraization of probability

$$P\colon A \to [0,1]$$

**Issues**

- It is not clear which structure on the co-domain $[0,1]$ is relevant.
- Which algebras should be in the domain / co-domain?
- The defining property of probability $P$ is not equational.
- Composition of probabilities is not defined.
- Can we make universal constructions work in probability theory?

# Outline

We introduce a 2-sorted algebraic framework for probability:

- We will define a probability algebra as a 2-sorted algebra

$$(M, N, p)$$

Events

Probability degrees

where $M, N$ are MV-algebras and

$$p \colon M \to N$$

is a probability map.

- The class of all algebras $(M, N, p)$ forms a 2-sorted variety.
- We characterize the free algebra.

# MV-algebras

An MV-algebra is essentially an order unit interval $[0, u]$ in a unital Abelian $\ell$-group $(G, u)$, endowed with the bounded operations of $G$.

MV-algebras form an equationally-defined class.

**Standard MV-algebra $[0, 1]_{MV}$**

$a \oplus b := \min(a + b, 1), \quad \neg a := 1 - a, \quad a \odot b := \max(a + b - 1, 0)$

**Free $n$-generated MV-algebra**

The algebra of continuous functions $[0, 1]^n \to [0, 1]$ that are

- piecewise linear and
- all linear pieces have $\mathbb{Z}$ coefficients.

## Probability maps

**Definition**

Let $M$ and $N$ be MV-algebras. A probability map is a function $p\colon M \to N$ such that for every $a, b \in M$ the following hold.

1. $p(a \oplus b) = p(a) \oplus p(b \wedge \neg a)$
2. $p(\neg a) = \neg p(a)$
3. $p(1) = 1$

- MV-homomorphisms $M \to N$
- Finitely-additive probability measures $B \to [0, 1]$
- Mundici's states $M \to [0, 1]$
- Flaminio-Montagna's internal states $M \to M$

## Example 1: Non-Archimedean co-domain

The Boolean algebra for a uniformly random selection of $n \in \mathbb{N}$ is

$$\mathcal{B} := \{A \subseteq \mathbb{N} \mid \text{either } A \text{ or } \neg A \text{ is finite}\}.$$

**Finitely-additive probability measure $\mathcal{B} \to [0, 1]$**

$$P(A) := \begin{cases} 0 & A \text{ finite,} \\ 1 & A \text{ cofinite.} \end{cases}$$

# Example 1: Non-Archimedean co-domain

The Boolean algebra for a uniformly random selection of $n \in \mathbb{N}$ is

$$\mathcal{B} \coloneqq \{A \subseteq \mathbb{N} \mid \text{either } A \text{ or } \neg A \text{ is finite}\}.$$

**Finitely-additive probability measure $\mathcal{B} \to [0, 1]$**

$$P(A) \coloneqq \begin{cases} 0 & A \text{ finite}, \\ 1 & A \text{ cofinite}. \end{cases}$$

Replace the co-domain $[0, 1]$ with Chang's MV-algebra

$$C \coloneqq \{0, \varepsilon, 2\varepsilon, \dots, 1 - 2\varepsilon, 1 - \varepsilon, 1\}.$$

**Probability map $\mathcal{B} \to C$**

$$p(A) \coloneqq \begin{cases} |A|\varepsilon & A \text{ finite}, \\ 1 - |\neg A|\varepsilon & A \text{ cofinite}, \end{cases}$$

# Example 2: PL-embedding

Define the state space of $M$:

$$\mathrm{St}\, M := \{s \colon M \to [0,1] \mid s \text{ is a state}\}$$

- For any $a \in M$, let $\bar{a} \colon \mathrm{St}\, M \to [0,1]$ be given by

$$\bar{a}(s) := s(a), \quad s \in \mathrm{St}\, M.$$

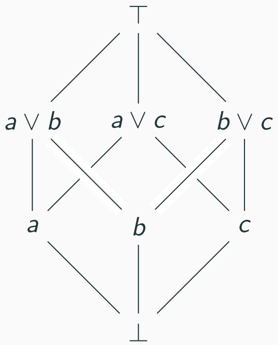
- Let $\nabla(M)$ be the MV-algebra generated by $\{\bar{a} \mid a \in M\}$.

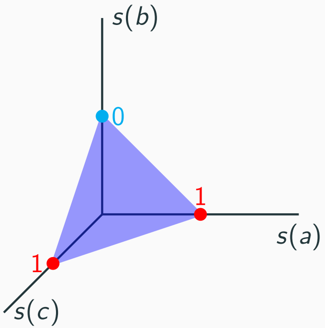
**Definition**

PL-embedding of $M$ is a probability map $\pi \colon M \to \nabla(M)$ given by

$$\pi(a) := \bar{a}, \quad a \in M.$$

# PL-embedding of a finite Boolean algebra




$a \vee c \in M \qquad \mapsto \qquad$ affine function $\overline{a \vee c}(s) \in \nabla(M)$

# Universal probability maps

**Theorem**

*For any MV-algebra $M$ there exists an MV-algebra $U(M)$ and a probability map*

$$\alpha \colon M \to U(M)$$

*such that $\alpha$ is universal (for $M$): for any probability map $p \colon M \to N$ there is exactly one MV-homomorphism $h \colon U(M) \to N$ satisfying*

$$\begin{tikzcd} M \arrow[rr, "\alpha"] \arrow[dr, "\text{probability map } p"'] & & U(M) \arrow[dl, dashed, "\text{MV-homomorphism } h"] \\ & N & \end{tikzcd}$$

# Universal probability maps

**Theorem**

*For any MV-algebra $M$ there exists an MV-algebra $U(M)$ and a probability map*

$$\alpha\colon M \to U(M)$$

*such that $\alpha$ is universal (for $M$): for any probability map $p\colon M \to N$ there is exactly one MV-homomorphism $h\colon U(M) \to N$ satisfying*

$$\begin{tikzcd} M \arrow[rr, "\alpha"] \arrow[dr, "\text{probability map } p"'] & & U(M) \arrow[dl, dashed, "\text{MV-homomorphism } h"] \\ & N & \end{tikzcd}$$

$M$ is semisimple *iff* $\alpha$ is the PL embedding $\pi$ of $M$

# Probability algebra

We introduce this two-sorted similarity type:

(T1) The single-sorted operations of MV-algebras $\oplus, \neg, 0$ in the 1st sort.

(T2) The single-sorted operations of MV-algebras $\oplus, \neg, 0$ in the 2nd sort.

(T3) The two-sorted operation $p$ between the two sorts.

# Probability algebra

We introduce this two-sorted similarity type:

(T1) The single-sorted operations of MV-algebras $\oplus, \neg, 0$ in the 1st sort.

(T2) The single-sorted operations of MV-algebras $\oplus, \neg, 0$ in the 2nd sort.

(T3) The two-sorted operation $p$ between the two sorts.

**Definition**

A probability algebra is an algebra $(M, N, p)$ of the two-sorted similarity type (T1)–(T3) such that

- $(M, \oplus, \neg, 0)$ is an MV-algebra.
- $(N, \oplus, \neg, 0)$ is an MV-algebra.
- The operation $p \colon M \to N$ is a probability map.

# Homomorphisms

A homomorphism between $(M_1, N_1, p_1)$ and $(M_2, N_2, p_2)$ is a function

$$h := (h_1, h_2)\colon\ (M_1, N_1) \to (M_2, N_2),$$

where $h_1\colon M_1 \to M_2$ and $h_2\colon N_1 \to N_2$ are MV-homomorphisms such that the following diagram commutes:

$$\begin{tikzcd} M_1 \arrow[r, "p_1"] \arrow[d, "h_1"'] & N_1 \arrow[d, "h_2"] \\ M_2 \arrow[r, "p_2"'] & N_2 \end{tikzcd}$$

# Free probability algebra

**Definition**

$$\begin{tikzcd} (S_1, S_2) \arrow[rr, "\iota"] \arrow[dr, "\eta"'] & & F(S_1, S_2) \arrow[dl, dashed, "h"] \\ & (M', N', p') & \end{tikzcd}$$

- By 2-sorted universal algebra $F(S_1, S_2)$ exists
- By category theory: since $(S_1, S_2) = S_1 \amalg S_2$ we get

$$F(S_1, S_2) = F(S_1, \emptyset) \amalg F(\emptyset, S_2)$$

# Free algebra generated by $(\emptyset, S_2)$

Let $(\emptyset, S_2)$ be a two-sorted set.

The probability algebra freely generated by $(\emptyset, S_2)$ is

$$(\mathbf{2}, F(S_2), p_0)$$

$\{0, 1\}$

The free MV-algebra over $S_2$

where $p_0$ is the unique probability map

$$\mathbf{2} \to F(S_2).$$

# Free algebra generated by $(S_1, \emptyset)$

Using the construction of universal probability map we get

**Theorem**

*Let $(S_1, \emptyset)$ be a two-sorted set of generators. Then the probability algebra freely generated by $(S_1, \emptyset)$ is*

$$(F(S_1),\ \nabla(F(S_1)),\ \pi),$$

*where*

$$\pi\colon F(S_1) \to \nabla(F(S_1))$$

*is the PL-embedding of the free MV-algebra $F(S_1)$.*

# Free algebra generated by $(S_1, S_2)$

**Theorem**

*Let $(S_1, S_2)$ be a two-sorted set. The probability algebra freely generated by $(S_1, S_2)$ is*

$$F(S_1, S_2) = (F(S_1),\ \nabla(F(S_1))\ \amalg_{MV}\ F(S_2),\ \tau),$$

*for $\tau := \beta_1 \circ \pi$, where*

$$F(S_1) \xrightarrow{\pi} \nabla(F(S_1)) \xrightarrow{\beta_1} \nabla(F(S_1))\ \amalg_{MV}\ F(S_2)$$

- *$\pi$ is the PL-embedding and*
- *$\beta_1$ is the coproduct injection.*

## Final remarks

$$\begin{array}{c}\text{MV-algebras} : \ell\text{-groups} \\ \simeq \\ \text{probability maps} : \text{unital positive group homomorphisms}\end{array}$$

- The total ignorance of an agent is modeled by the universal map

$$M \xrightarrow{\alpha} U(M)$$

- Is $F(S_1, S_2)$ semisimple?
- Independence and conditioning for probability maps/algebras