# Disjunction and Existence Properties in Inquisitive Logic

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June 30, 2017

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## Motivating example: hospital protocol

- A disease gives rise to two symptoms $S_1$ and $S_2$.
- $S_1$ is much worse than $S_2$.
- Depending on which symptoms the patients show, they have to be put in quarantine.

# Motivating example: hospital protocol

- A disease gives rise to two symptoms $S_1$ and $S_2$.
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- Depending on which symptoms the patients show, they have to be put in quarantine.

## Protocol

- Patient $x$ shows $S_1 \Rightarrow x$ in quarantine.
- Everyone shows $S_2 \Rightarrow$ Everyone in quarantine.
- Otherwise, no quarantine.

$$\begin{aligned} Q_1: &\quad \textit{Wether x shows } S_1 \\ Q_2: &\quad \textit{Wether everyone shows } S_2 \\ &\quad \textbf{determine} \\ Q_3: &\quad \textit{Wether x is in quarantine} \end{aligned}$$

$$\begin{array}{ll} Q_1: & \textit{Wether } x \textit{ shows } S_1 \\ Q_2: & \textit{Wether everyone shows } S_2 \\ & \textbf{determine} \\ Q_3: & \textit{Wether } x \textit{ is in quarantine} \end{array}$$

<u>Observation:</u> $Q_1$, $Q_2$ and $Q_3$ are *questions*.

**Question** $Q_3$ *depends* on **questions** $Q_1$ and $Q_2$.

*How can we represent*

***dependency between questions***

*in a logical framework?*

**Question** $Q_3$ *depends* on **questions** $Q_1$ and $Q_2$.

## Logic and Questions

In `FOL` (classical first-order logic) a formula is determined by its associated truth-value in any context $\Rightarrow$ a `FOL` formula represents a **statement**.

**Questions** do not have an associated truth-value $\Rightarrow$ questions are not (directly) representable in `FOL`.

The aim of the logic `InqBQ` (inquisitive first-order logic) is to

- extend `FOL` to represent **questions** as formulas;
- extend `FOL` entailment to capture **dependency** between questions.

`InqBQ`: Adding Questions to `FOL`

Disjunction Property

Existence Property

# InqBQ: **Adding Questions to** FOL

# Syntax of `InqBQ`: introducing questions

$$\phi ::= \bot \mid [t_1 = t_2] \mid R(t_1, \ldots, t_n) \mid \phi \wedge \phi \mid \phi \to \phi \mid \forall x.\phi \quad \mid \quad \phi \mathbin{\mathpalette\make@circled\vee} \phi \mid \overline{\exists}\, x.\phi$$

**shorthands**

$\neg\phi := \phi \to \bot$ $\qquad$ $\phi \vee \psi := \neg(\neg\phi \wedge \neg\psi)$ $\qquad$ $\exists x.\phi := \neg\forall x.\neg\phi$

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A formula is called `FOL` or *classical* if it does not contain the symbols $\vee\!\!\!\vee$ and $\overline{\exists}$.

`FOL` formulas are denoted with $\alpha$, $\beta$, ...

## Intuition

`FOL` formulas represent *statements*.

$$(c = d) \vee (c \neq d) \quad \equiv \quad \text{“}c \textit{ is equal to } d \textit{ or not”}$$

$$\exists x.[x = c] \quad \equiv \quad \textit{“There is an element equal to } c\text{”}$$

The operator $\mathbin{\vee\!\!\!\vee}$ introduces *alternative questions*.

$$(c = d) \mathbin{\vee\!\!\!\vee} (c \neq d) \quad \equiv \quad \textit{“Is } c \textit{ equal to } d \textit{ or not?”}$$

The operator $\overline{\exists}$ introduces *existential questions*.

$$\overline{\exists}\, x.[x = c] \quad \equiv \quad \textit{“Which is an element equal to } c\textit{?”}$$

# Some notations

Fix a signature $\Sigma = \{f_i, R_j\}_{i \in I, j \in J}$.

**Definition (`FOL` structure)**

$$M = \langle\, D\,,\, \mathbf{f}_i\,,\, \mathbf{R}_j\,,\, \sim \rangle_{i \in I, j \in J}$$

where

- $\mathbf{f}_i : D^{\mathrm{ar}(f_i)} \to D$ is the interpretation of $f_i$;
- $\mathbf{R}_j \subseteq D^{\mathrm{ar}(R_j)}$ is the interpretation of $R_j$;
- $[\sim] \subseteq D^2$ is an equivalence relation and a congruence with respect to $\{\mathbf{f}_i\,,\, \mathbf{R}_j\}_{i \in I, j \in J}$.

$$M = \langle\, D\,,\, \mathbf{f}_i\,,\, \mathbf{R}_j\,,\, \sim \rangle_{i\in I, j\in J}$$

**Definition (Skeleton)**

Given $M$ a `FOL` structure, define

$$\mathrm{Sk}(M) = \langle D, \mathbf{f}_i \rangle_{i\in I}$$

i.e., leaving out relations and equality.

# Models of `InqBQ`: representing information

**Definition (Information structure)**

$$\mathcal{M} = \left\langle M_w | w \in W^{\mathcal{M}} \right\rangle$$

where the $M_w$ are classical structures sharing *the same skeleton*.

We will call $W^{\mathcal{M}}$ the set of *worlds* of the structure.

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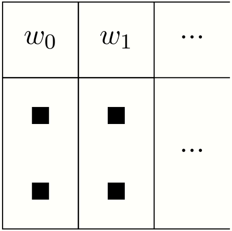


Example of a simple model in the signature $\{\mathbf{f}^{(1)}\}$.

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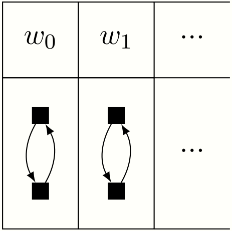


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The arrow represents $\mathbf{f}$.

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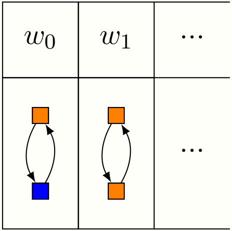
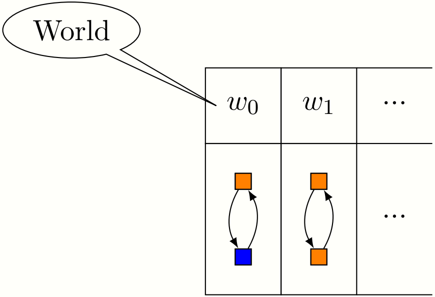


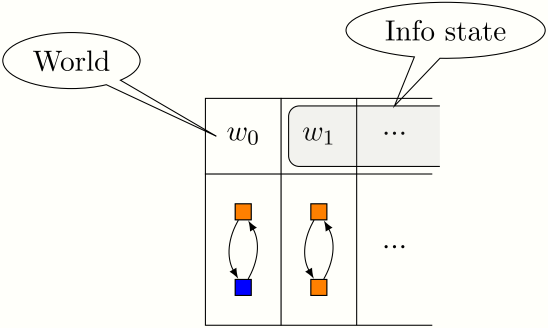

Example of a simple model in the signature $\{\mathbf{f}^{(1)}\}$.

The arrow represents $\mathbf{f}$. The colours represent equality.

| $w_0$ | $w_1$ | $\cdots$ |
|---|---|---|
|  |  | $\cdots$ |




**Truth-condition** encoded by **World**




| | | |
|---|---|---|
| **Truth-condition** | encoded by | **World** |
| **Information** | encoded by | **Info State** |

# Semantics of `InqBQ`: supporting relation

$\mathcal{M} \rightsquigarrow$ info structure $\quad s \rightsquigarrow$ info state $\quad g \rightsquigarrow$ assignment

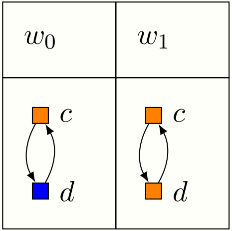
$$\mathcal{M}, s \vDash_g \phi$$

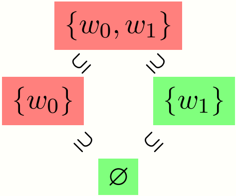
$$\begin{array}{lll}
\mathcal{M}, s \vDash_g \bot & \iff & s = \varnothing \\
\mathcal{M}, s \vDash_g [t_1 = t_2] & \iff & \forall w \in s. \left[ g(t_1) \sim_w^{\mathcal{M}} g(t_2) \right] \\
\mathcal{M}, s \vDash_g R(t_1, \ldots, t_n) & \iff & \forall w \in s. \left[ \mathbf{R}_w^{\mathcal{M}}(g(t_1), \ldots, g(t_n)) \right] \\
\mathcal{M}, s \vDash_g \phi \wedge \psi & \iff & \mathcal{M}, s \vDash_g \phi \text{ and } \mathcal{M}, s \vDash_g \psi \\
\mathcal{M}, s \vDash_g \phi \to \psi & \iff & \forall t \subseteq s. [\mathcal{M}, t \vDash_g \phi \Rightarrow \mathcal{M}, t \vDash_g \psi] \\
\mathcal{M}, s \vDash_g \forall x.\phi & \iff & \forall d \in D^{\mathcal{M}}. \mathcal{M}, s \vDash_{g[x \mapsto d]} \phi
\end{array}$$

$$\begin{array}{lll}
\mathcal{M}, s \vDash_g \phi \mathbin{\vee\!\!\!\vee} \psi & \iff & \mathcal{M}, s \vDash_g \phi \text{ or } \mathcal{M}, s \vDash_g \psi \\
\mathcal{M}, s \vDash_g \overline{\exists} x.\phi & \iff & \exists d \in D^{\mathcal{M}}. \mathcal{M}, s \vDash_{g[x \mapsto d]} \phi
\end{array}$$

$$\mathcal{M}, s \vDash_g [t_1 = t_2] \iff \forall w \in s. \left[ g(t_1) \sim^{\mathcal{M}}_{w} g(t_2) \right]$$

$$c = d \quad \cong \quad \text{“}c \textit{ is equal to } d\text{”}$$



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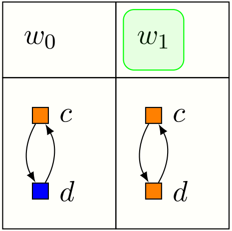


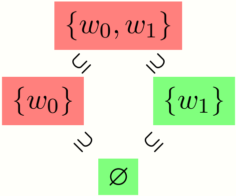

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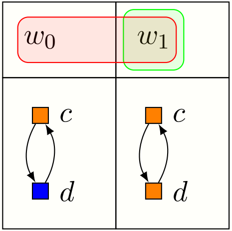


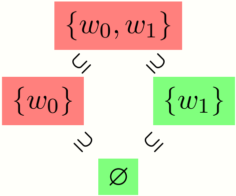

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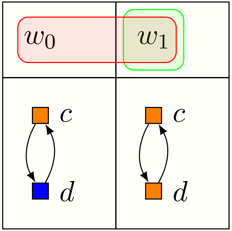


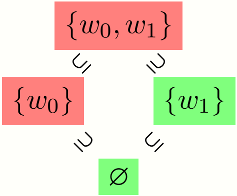


**Fact 1:** The info states that support a `FOL` formula form a *principal ideal* (truth-conditionality).

An alternative way to state this: $s \vDash \alpha$ iff $\forall w \in s. \{w\} \vDash \alpha$.

$$\mathcal{M}, s \vDash_g \phi \mathbin{\vee\!\!\!\vee} \psi \quad \Longleftrightarrow \quad \mathcal{M}, s \vDash_g \phi \text{ or } \mathcal{M}, s \vDash_g \psi$$

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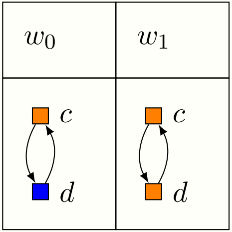


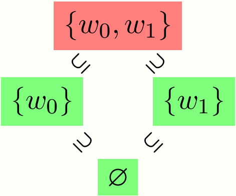

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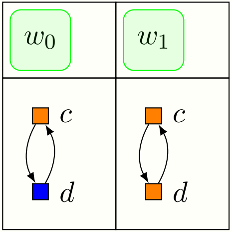


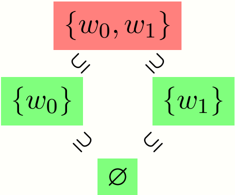

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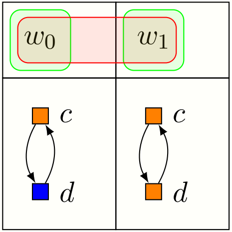


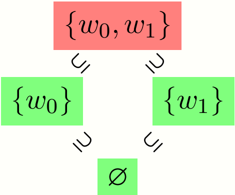

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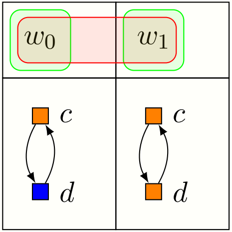
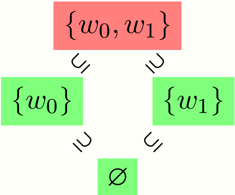


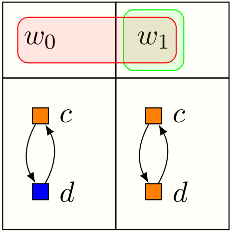

**Fact 2:** The info states that support a formula form an *ideal*, but in general *not principal* (Persistency).

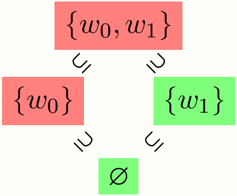
Uniform substitution does not hold!

**Fact 3:** $\phi$ is truth-conditional iff is equivalent to a `FOL` formula.

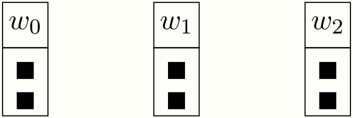
$$\mathcal{M}, s \vDash_g \overline{\exists}\, x.\phi \quad \Longleftrightarrow \quad \exists d \in D^{\mathcal{M}}.\, \mathcal{M}, s \vDash_{g[x \mapsto d]}$$

$$\overline{\exists}\, x.[f(x) = x] \quad \equiv \quad \text{“\textit{Which is a fixed point of } } f\text{\textit{?}”}$$



## Some insight. . . Information structures as Kripke models



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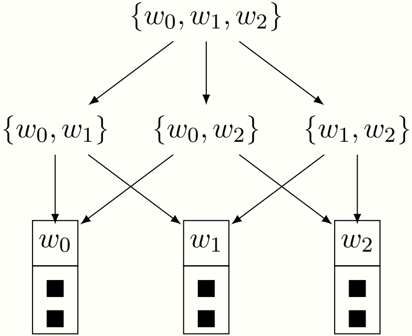

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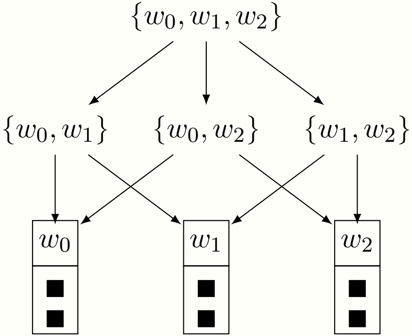


- $\text{Frame} = \langle \mathcal{P}(W) \setminus \{\varnothing\}, \supseteq \rangle$

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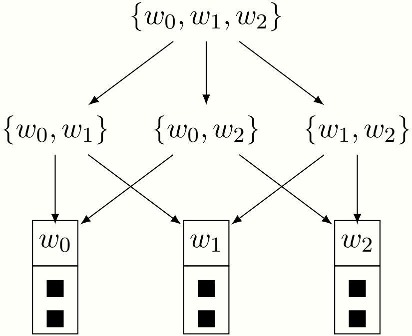


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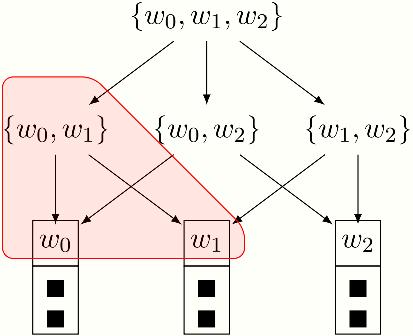


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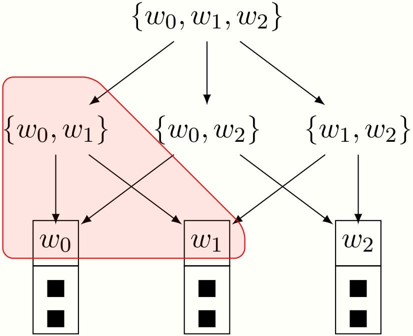


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**Fact:** `InqBQ` is the logic of a class of *Kripke models*.

# The Main Result: DP and EP in `InqBQ`

**Theorem (Disjunction and Existence Property)**

*Consider* $\Gamma$ *a* `FOL` *theory. Then*

- *If* $\Gamma \vDash \phi \mathbin{\mathbin{\vee}\!\!\!\vee} \psi$ *then* $\Gamma \vDash \phi$ *or* $\Gamma \vDash \psi$.
- *If* $\Gamma \vDash \overline{\exists}\, x.\phi(x)$ *then* $\Gamma \vDash \phi(t)$ *for some term* $t$.

**Corollary**

If $\Gamma \vDash \forall \overline{x}\, \overline{\exists}! y.\phi(\overline{x}, y)$ (i.e., $\phi$ defines a function), then there exists a term $t$ such that $\Gamma \vDash \forall \overline{x}.\phi(\overline{x}, t)$.

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**But how do we prove this?**

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**But how do we prove this?**

By playing with the models!

# Model-theoretic constructions

$$
\begin{tikzcd}
\mathcal{B}\mathcal{M} \arrow[dr] & & \mathcal{M}^{\sigma} \arrow[dl] & & & & & \\
& \mathcal{B}^{\sigma}\mathcal{M} \arrow[drr] & & \mathcal{M}_{\omega} \arrow[d] & & \mathcal{M} \uplus \mathcal{N} \arrow[dll] \arrow[dr] & & M \star S \arrow[dl] \\
& & & \mathfrak{S}\,\mathcal{M} \arrow[ddr] & & & \mathcal{M} \oplus \mathcal{N} \arrow[d] & \\
& & & & & & \mathcal{M}_{\Gamma} \arrow[dll] & \\
& & & & \mathfrak{S}(\mathcal{M}_{\Gamma}) & & &
\end{tikzcd}
$$

# Disjunction Property

# Disjunction Property - Proof idea

$$\Gamma \not\models \phi \text{ and } \Gamma \not\models \psi \implies \Gamma \not\models \phi \mathbin{\veebar} \psi$$

$\mathcal{M}$

$$\boxed{\begin{array}{c} \models \Gamma \\ \not\models \phi \\ \models \psi \end{array}}$$

$\mathcal{N}$

$$\boxed{\begin{array}{c} \models \Gamma \\ \models \phi \\ \not\models \psi \end{array}}$$

# Disjunction Property - Proof idea

$$\Gamma \nvDash \phi \text{ and } \Gamma \nvDash \psi \implies \Gamma \nvDash \phi \Vee \psi$$

$$\begin{tikzcd}
 & \boxed{\begin{array}{l} \nvDash \psi \\ \nvDash \phi \\ \vDash \Gamma \end{array}} & \\
\mathcal{M} \arrow[r, hook] & \mathcal{M} \oplus \mathcal{N} & \mathcal{N} \arrow[l, hook'] \\
\boxed{\begin{array}{l} \vDash \Gamma \\ \nvDash \phi \\ \vDash \psi \end{array}} & & \boxed{\begin{array}{l} \vDash \Gamma \\ \vDash \phi \\ \nvDash \psi \end{array}}
\end{tikzcd}$$

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$$\begin{tikzcd}
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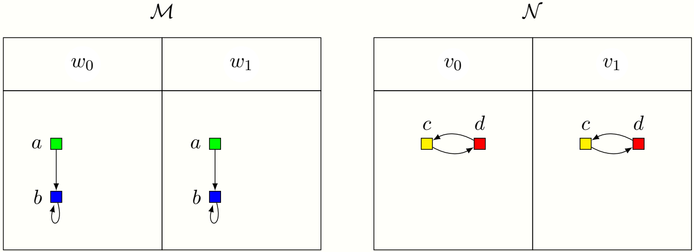
\boxed{\begin{array}{c} \color{red}{\vDash \Gamma} \\ \nvDash \phi \\ \vDash \psi \end{array}} & & \boxed{\begin{array}{c} \color{red}{\vDash \Gamma} \\ \vDash \phi \\ \nvDash \psi \end{array}}
\end{tikzcd}$$

# Combining models - the direct sum $\oplus$

$$
\begin{tikzcd}
\mathcal{B}\mathcal{M} \arrow[dr] & & \mathcal{M}^{\sigma} \arrow[dl] & & & & & \\
& \mathcal{B}^{\sigma}\mathcal{M} \arrow[drr] & & \mathcal{M}_{\omega} \arrow[d] & & \mathcal{M} \uplus \mathcal{N} \arrow[dll] \arrow[dr] & & M \star S \arrow[dl] \\
& & & \mathfrak{S}\,\mathcal{M} \arrow[ddr] & & & \mathcal{M} \oplus \mathcal{N} \arrow[d] & \\
& & & & & & \mathcal{M}_{\Gamma} \arrow[dll] & \\
& & & & \mathfrak{S}(\mathcal{M}_{\Gamma}) & & &
\end{tikzcd}
$$

We can define a model $\mathcal{M} \oplus \mathcal{N}$ such that

$$W^{\mathcal{M} \oplus \mathcal{N}} = W^{\mathcal{M}} \sqcup W^{\mathcal{N}} \qquad \text{and} \qquad D^{\mathcal{M} \oplus \mathcal{N}} = D^{\mathcal{M}} \times D^{\mathcal{N}}$$



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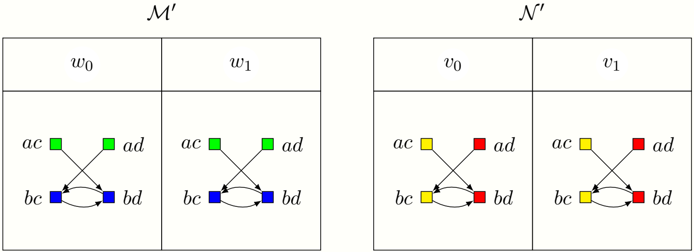


$$\mathbf{f}^{\mathcal{M}'}(ac) = \left\langle \mathbf{f}^{\mathcal{M}}(a), \mathbf{f}^{\mathcal{N}}(c) \right\rangle$$

$$\langle x, y \rangle \sim_{w_0}^{\mathcal{M}'} \langle x', y' \rangle \iff x \sim_{w_0}^{\mathcal{M}} x'$$

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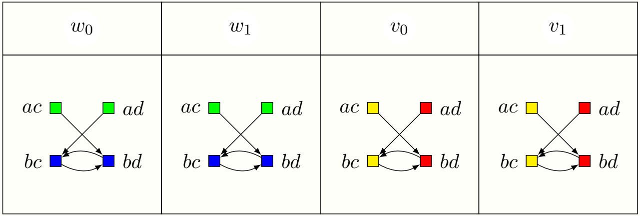

**Theorem (Main property of $\oplus$)**

*Let $s \subseteq W^{\mathcal{M}}$, $g : \mathrm{Var} \to D^{\mathcal{M}} \times D^{\mathcal{N}}$ an assignment, $\phi$ a formula. Then:*

$$\mathcal{M} \oplus \mathcal{N}, s \vDash_g \phi \iff \mathcal{M}, s \vDash_{\pi_1 g} \phi$$

**Corollary**

- Let $\Gamma$ be a `FOL` theory. If $\mathcal{M} \vDash_{\pi_1 g} \Gamma$ and $\mathcal{N} \vDash_{\pi_2 g} \Gamma$ then $\mathcal{M} \oplus \mathcal{N} \vDash_g \Gamma$.
- Let $\phi$ be a formula. If $\mathcal{M} \nvDash_{\pi_1 g} \phi$ then $\mathcal{M} \oplus \mathcal{N} \nvDash_g \phi$.

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- Let $\phi$ be a formula. If $\mathcal{M} \nvDash_{\pi_1 g} \phi$ then $\mathcal{M} \oplus \mathcal{N} \nvDash_g \phi$.

And this is exactly what we needed!

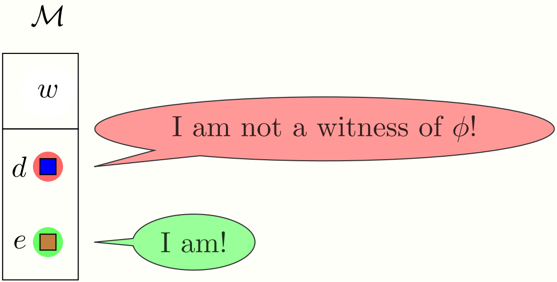
**Corollary**
A `FOL` theory $\Gamma$ has the disjunction property.

# Existence Property

# Existence Property - Proof Strategy

$$\Gamma \not\models \phi(t) \text{ for all } t \implies \Gamma \not\models \overline{\exists}\, x.\phi(x)$$

## Strategy



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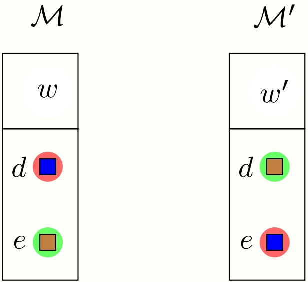

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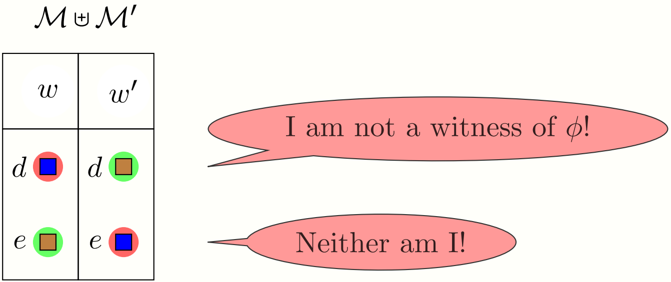

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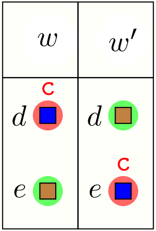

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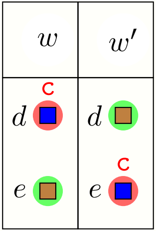



We need a way to deal with the interpretation of the functions.

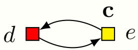
# Relaxing the structure - the blow up model $\mathcal{BM}$

$$
\begin{tikzcd}
\mathcal{BM} \arrow[dr] & & \mathcal{M}^\sigma \arrow[dl] & & & & & \\
& \mathcal{B}^\sigma\mathcal{M} \arrow[drr] & & \mathcal{M}_\omega \arrow[d] & & \mathcal{M} \uplus \mathcal{N} \arrow[dll] \arrow[dr] & & M \star S \arrow[dl] \\
& & & \mathfrak{S}\,\mathcal{M} \arrow[ddr] & & & \mathcal{M} \oplus \mathcal{N} \arrow[d] & \\
& & & & & & \mathcal{M}_\Gamma \arrow[dll] & \\
& & & & \mathfrak{S}(\mathcal{M}_\Gamma) & & &
\end{tikzcd}
$$

We want to define a model $\mathcal{BM}$ *elementary equivalent* to $\mathcal{M}$ such that

$$W^{\mathcal{BM}} = W^{\mathcal{M}} \qquad D^{\mathcal{BM}} = \{\text{closed terms of } \Sigma(D^{M})\}$$

$$\Sigma = \left\{c; f^{(1)}\right\}$$



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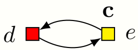


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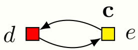


| | | |
|---|---|---|
| $\vdots$ | $\vdots$ | $\vdots$ |
| $f(f(\underline{d}))$ | $f(f(\underline{e}))$ | $f(f(c))$ |
| $f(\underline{d})$ | $f(\underline{e})$ | $f(c)$ |
| $\underline{d}$ | $\underline{e}$ | $c$ |

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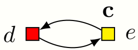


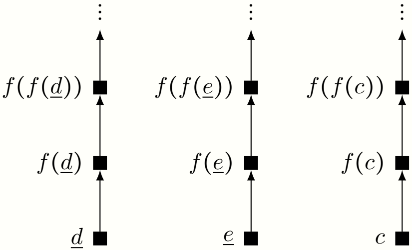


$$\mathbf{f}^{\mathcal{BM}}(t) = f(t)$$

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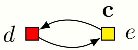
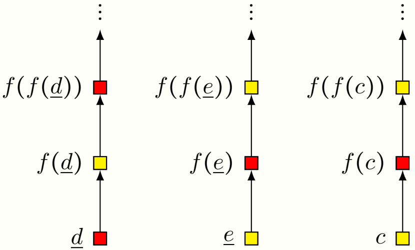


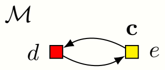

$$\mathbf{f}^{\mathcal{BM}}(t) = f(t) \qquad t_1 \sim^{\mathcal{BM}} t_2 \iff t_1^{\mathcal{M}} \sim^{\mathcal{M}} t_2^{\mathcal{M}}$$

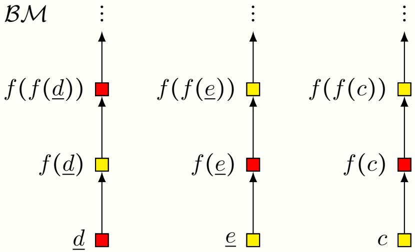
**Theorem (Blow-up main property)**

*Let* $s \subseteq W^{\mathcal{M}}$ *be an info state,* $t_1, \ldots, t_n$ *closed terms of* $\Sigma\left(D^{\mathcal{M}}\right)$ *and* $\phi(x_1, \ldots, x_n)$ *a formula. Then*

$$\mathcal{B}\mathcal{M}, s \vDash \phi\,(t_1, \ldots, t_n) \iff \mathcal{M}, s \vDash \phi\left(t_1^{\mathcal{M}}, \ldots, t_n^{\mathcal{M}}\right)$$

Now that we relaxed the structure, we can permute the elements of $\mathcal{M}$ preserving the skeleton.



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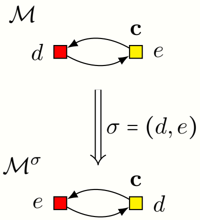


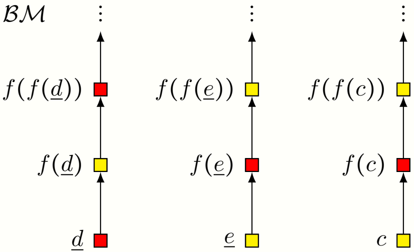

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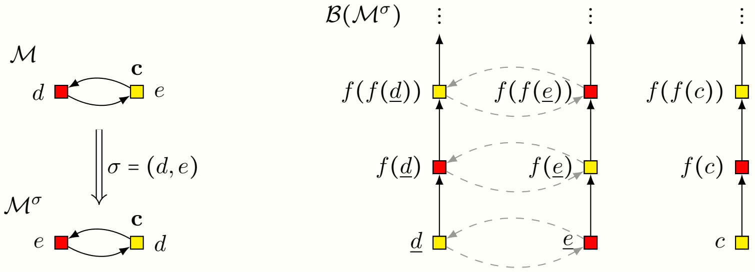

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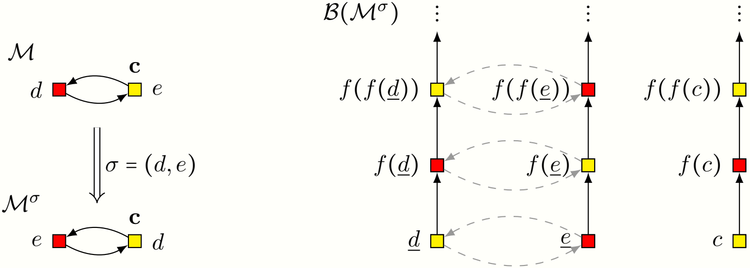


The role of the elements $\underline{d}$ and $\underline{e}$ has been reversed, while $c$ assumes the same role.

# Swapping and gluing - full permutation model $\mathfrak{S}\,\mathcal{M}$

$$
\begin{tikzcd}
\mathcal{B}\mathcal{M} \arrow[dr] & & \mathcal{M}^{\sigma} \arrow[dl] & & & & & \\
& \mathcal{B}^{\sigma}\mathcal{M} \arrow[drr] & & \mathcal{M}_{\omega} \arrow[d] & & \mathcal{M} \uplus \mathcal{N} \arrow[dll] \arrow[dr] & & M \star S \arrow[dl] \\
& & & {\color{red}\mathfrak{S}\,\mathcal{M}} \arrow[ddr] & & & \mathcal{M} \oplus \mathcal{N} \arrow[d] & \\
& & & & & & \mathcal{M}_{\Gamma} \arrow[dll] & \\
& & & & \mathfrak{S}(\mathcal{M}_{\Gamma}) & & &
\end{tikzcd}
$$

$$\mathcal{M} \quad \rightsquigarrow \quad \mathcal{B}\mathcal{M} \quad \rightsquigarrow \quad \mathcal{B}^{\sigma}\mathcal{M} \quad \rightsquigarrow \quad \mathfrak{S}\,\mathcal{M}$$

**The full permutation model** - $\mathfrak{S}\,\mathcal{M}$

The idea to build up the model $\mathfrak{S}\,\mathcal{M}$ is to consider all the models $\mathcal{B}^{\sigma}\mathcal{M}$ for $\sigma \in \mathfrak{S}\left(D^{\mathcal{M}}\right)$ and combine them into a unique structure. This is possible because the models $\mathcal{B}^{\sigma}\mathcal{M}$ *share the same skeleton*.

$$\mathcal{M} \quad \rightsquigarrow \quad \mathcal{B}\mathcal{M} \quad \rightsquigarrow \quad \mathcal{B}^{\sigma}\mathcal{M} \quad \rightsquigarrow \quad \mathfrak{S}\,\mathcal{M}$$

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**Theorem (Properties of $\mathfrak{S}\,\mathcal{M}$)**

- *Let* $\Gamma$ *be a* `FOL` *theory. If* $\mathcal{M} \vDash \Gamma$ *then* $\mathfrak{S}\,\mathcal{M} \vDash \Gamma$.
- *Let* $g$ *be a fixed assignment. If* $\mathcal{M} \nvDash_g \phi(t)$ *for every term* $t$, *then* $\mathfrak{S}\,\mathcal{M} \nvDash \overline{\exists}\, x.\phi(x)$.

# The characteristic model of a FOL theory - $\mathcal{M}_\Gamma$

$$
\begin{tikzcd}
\mathcal{BM} \arrow[dr] & & \mathcal{M}^\sigma \arrow[dl] & & & & & \\
& \mathcal{B}^\sigma\mathcal{M} \arrow[drr] & & \mathcal{M}_\omega \arrow[d] & & \mathcal{M} \uplus \mathcal{N} \arrow[dll] \arrow[dr] & & M \star S \arrow[dl] \\
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& & & & & & {\color{red}\mathcal{M}_\Gamma} \arrow[dll] & \\
& & & & \mathfrak{S}(\mathcal{M}_\Gamma) & & &
\end{tikzcd}
$$

**Theorem (The characteristic model of $\Gamma$)**

*Given $\Gamma$ a* `FOL` *theory, there exists a model $\mathcal{M}_\Gamma$ and an evaluation $g_\Gamma$ such that*

$$\mathcal{M}_\Gamma \models_{g_\Gamma} \phi \iff \Gamma \models \phi$$

**Idea to build $\mathcal{M}_\Gamma$**

- For every non-entailment $\Gamma \not\models \psi$ choose $\left\langle \mathcal{M}_\psi, g_\psi \right\rangle$ such that

$$\mathcal{M}_\psi \models \Gamma \qquad \mathcal{M}_\psi \not\models_{g_\psi} \psi$$

- Combine the models and assignments choosen.

# Existence property - proof

**Theorem**

*Let* $\Gamma$ *be a closed* `FOL` *theory. Then*

$$\Gamma \not\models \phi(t) \textit{ for every } t \textit{ term} \implies \Gamma \not\models \overline{\exists}\, x.\phi(x)$$

**Proof**

Consider the characteristic model $\mathcal{M}_\Gamma$ and the assignment $g_\Gamma$. Then

$$\begin{aligned} \mathcal{M}_\Gamma \models \Gamma \quad & \implies \mathfrak{S}(\mathcal{M}_\Gamma) \models \Gamma \\ \mathcal{M}_\Gamma \not\models_{g_\Gamma} \phi(t) \text{ for every } t \quad & \implies \mathfrak{S}(\mathcal{M}_\Gamma) \not\models \overline{\exists}\, x.\phi(x) \end{aligned}$$

Thus $\Gamma \not\models \overline{\exists}\, x.\phi(x)$ as wanted.

**Thank you for your attention!**

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**Definition (Support semantics)**

Let $\mathcal{M} = \left\langle M_w | w \in W^{\mathcal{M}} \right\rangle$ be a model, $s \subseteq W^{\mathcal{M}}$ an info state and $g : \mathrm{Var} \to D^{\mathcal{M}}$ an assignment. We define

$$
\begin{array}{lcl}
\mathcal{M}, s \vDash_g \bot & \iff & s = \varnothing \\
\mathcal{M}, s \vDash_g [t_1 = t_2] & \iff & \forall w \in s. \left[ g(t_1) \sim_w^{\mathcal{M}} g(t_2) \right] \\
\mathcal{M}, s \vDash_g R(t_1, \ldots, t_n) & \iff & \forall w \in s. \left[ \mathbf{R}_w^{\mathcal{M}}(g(t_1), \ldots, g(t_n)) \right] \\
\mathcal{M}, s \vDash_g \phi \wedge \psi & \iff & \mathcal{M}, s \vDash_g \phi \text{ and } \mathcal{M}, s \vDash_g \psi \\
\mathcal{M}, s \vDash_g \phi \to \psi & \iff & \forall t \subseteq s. [\mathcal{M}, t \vDash_g \phi \Rightarrow \mathcal{M}, t \vDash_g \psi] \\
\mathcal{M}, s \vDash_g \forall x. \phi & \iff & \forall d \in D^{\mathcal{M}}. \mathcal{M}, s \vDash_{g[x \mapsto d]} \phi
\end{array}
$$

$$
\begin{array}{lcl}
\mathcal{M}, s \vDash_g \phi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi & \iff & \mathcal{M}, s \vDash_g \phi \text{ or } \mathcal{M}, s \vDash_g \psi \\
\mathcal{M}, s \vDash_g \overline{\exists} x. \phi & \iff & \exists d \in D^{\mathcal{M}}. \mathcal{M}, s \vDash_{g[x \mapsto d]} \phi
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\mathcal{M}, s \vDash_g \forall x.\phi & \iff & \forall d \in D^{\mathcal{M}}. \mathcal{M}, s \vDash_{g[x \mapsto d]} \phi
\end{array}
$$

$$
\begin{array}{lcl}
\mathcal{M}, s \vDash_g \phi \vee\!\!\!\vee \psi & \iff & \mathcal{M}, s \vDash_g \phi \text{ or } \mathcal{M}, s \vDash_g \psi \\
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**Definition (Support semantics)**

Let $\mathcal{M} = \left\langle M_w | w \in W^{\mathcal{M}} \right\rangle$ be a model, $s \subseteq W^{\mathcal{M}}$ an info state and $g : \mathrm{Var} \to D^{\mathcal{M}}$ an assignment. We define

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**Definition (Direct sum - $\oplus$)**

- $W^{\mathcal{M}\oplus\mathcal{N}} = W^{\mathcal{M}} \sqcup W^{\mathcal{N}}$
- $D^{\mathcal{M}\oplus\mathcal{N}} = D^{\mathcal{M}} \times D^{\mathcal{N}}$
- $f^{\mathcal{M}\oplus\mathcal{N}} = \left\langle f^{\mathcal{M}}; f^{\mathcal{N}} \right\rangle$
- If $w \in W^{\mathcal{M}}$ then $\langle d_1, e_1 \rangle \sim_w^{\mathcal{M}\oplus\mathcal{N}} \langle d_2, e_2 \rangle \iff d_1 \sim^{\mathcal{M}} d_2$
  If $w \in W^{\mathcal{N}}$ then $\langle d_1, e_1 \rangle \sim_w^{\mathcal{M}\oplus\mathcal{N}} \langle d_2, e_2 \rangle \iff e_1 \sim^{\mathcal{N}} e_2$
- If $w \in W^{\mathcal{M}}$ then
  $R_w^{\mathcal{M}\oplus\mathcal{N}}(\langle d_1, e_1 \rangle, \ldots, \langle d_n, e_n \rangle) = R_w^{\mathcal{M}}(d_1, \ldots, d_n)$
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**Definition (Blowup Model)**

Given a model $\mathcal{M}$ we define its **blow-up** as the model

$$\mathcal{BM} = \left\langle W^{\mathcal{M}},\ D^{\mathcal{BM}},\ I^{\mathcal{BM}},\ \sim^{\mathcal{BM}} \right\rangle$$

where

- $D^{\mathcal{BM}}$ is the set of terms in the signature $\Sigma \sqcup \{\underline{d} | d \in D^{\mathcal{M}}\}$
- Given $\underline{t_1}, \underline{t_2}, \dots \in D^{\mathcal{BM}}$ we define

$$\underline{t_1} \sim_w^{\mathcal{BM}} \underline{t_2} \iff t_1 \sim_w^{\mathcal{M}} t_2$$
$$R_w^{\mathcal{BM}}(\,\underline{t_1}, \dots, \underline{t_n}\,) \iff R_w^{\mathcal{M}}(\,t_1, \dots, t_n\,)$$

- $f^{\mathcal{BM}}$ is defined as the formal term combinator

$$f^{\mathcal{BM}}(\,\underline{t_1}, \dots, \underline{t_n}\,) = \underline{f(t_1, \dots, t_n)}$$

**Definition (The permutation model $\mathcal{B}_\sigma\mathcal{M}$)**

Given $\mathcal{M}$ a model and $\sigma \in \mathfrak{S}(D^{\mathcal{M}})$ a permutation, we define

$$\mathcal{B}_\sigma\mathcal{M} = \left\langle W^{\mathcal{M}},\ D^{\mathcal{B}\mathcal{M}},\ I^{\mathcal{B}_\sigma\mathcal{M}},\ \sim^{\mathcal{B}_\sigma\mathcal{M}} \right\rangle$$

where

- $f^{\mathcal{B}_\sigma\mathcal{M}} = f^{\mathcal{B}\mathcal{M}}$ is the formal combinator of terms.
- Given $\underline{t_1}, \underline{t_2}, \dots \in D^{\mathcal{B}\mathcal{M}}$ it holds

$$R_w^{\mathcal{B}_\sigma\mathcal{M}}\left(\underline{t_1}, \dots, \underline{t_n}\right) \iff R_w^{\mathcal{B}\mathcal{M}}\left(\sigma^{-1}\underline{t_1}, \dots, \sigma^{-1}\underline{t_n}\right)$$
$$\underline{t_1} \sim_w^{\mathcal{B}_\sigma\mathcal{M}} \underline{t_2} \iff \sigma^{-1}\underline{t_1} \sim_w^{\mathcal{B}\mathcal{M}} \sigma^{-1}\underline{t_2}$$

# Hospital protocol: formalization

The protocol:

$$\tau \equiv Q(x) \leftrightarrow S_1(x) \vee \forall y. S_2(y)$$

The dependence:

$$\tau,\ ?S_1(x) \mathbin{\vee\!\!\!\vee} ?\forall y. S_2(y) \vDash ?Q(x)$$

| $w_0$ | $w_1$ | $w_2$ |
|---|---|---|
| ■ $S_1, S_2, Q$ | ■ $S_2$ | ■ $S_1, S_2, Q$ |
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