Deductive systems

Definition
A logic is a consequence relation $\vdash$ on the set of formulas of an algebraic language built up with an infinite set of variables s.t.

if $\Gamma \vdash \phi$, then $\sigma(\Gamma) \vdash \sigma(\phi)$.

Let $\vdash$ be a logic, $A$ an algebra and $F \subseteq A$.

1. The Leibniz congruence $\Omega^A_F$ is the largest congruence $\theta$ of $A$ s.t. $F$ is a union of blocks of $\theta$.
2. The Suszko congruence is $\sim_{\Omega^A_F} := \bigcap \{ \Omega^A_G : G \in \mathcal{F}_I A \text{ and } F \subseteq G \}$.
3. The Suszko models of $\vdash$ are $\text{Mod}^\text{Su}(\vdash) := \{ \langle A, F \rangle : \sim_{\Omega^A_F} = \text{Id}_A \text{ and } F \in \mathcal{F}_I A \}$.

Interpretations between deductive systems

Definition
An interpretation $\tau$ of $\vdash$ into $\vdash'$ is a map that associates every basic $n$-ary connective $f(x_1, \ldots, x_n)$ of $\vdash$ to a term $\varphi(x_1, \ldots, x_n)$ of $\vdash'$ in such a way that

if $\langle A, F \rangle \in \text{Mod}^\text{Su}(\vdash')$, then $\langle A^\tau, F \rangle \in \text{Mod}^\text{Su}(\vdash)$

where $A^\tau := \langle A, \{ \tau(f) : f \text{ is a connective of } \vdash \} \rangle$.

Examples:
- The identity is an interpretation of $\text{IPC}$ in $\text{CPC}$.
- The identity is an interpretation of $\text{CPC}_{\land \lor}$ in $\text{CPC}$.

The poset of logics

Definition
We define a pre-order between logics as follows:

$\vdash \leq \vdash' \iff$ there is an interpretation of $\vdash$ into $\vdash'$.

Then we set

$\llbracket \vdash \rrbracket := \{ \vdash' : \vdash' \text{ is a logic equi-interpretable with } \vdash \}$.

Let $\text{Log}$ be the poset, whose elements are the classes $\llbracket \vdash \rrbracket$.

Theorem
$\text{Log}$ is a complete meet-semilattice, but it is not a join-semilattice. Moreover, $\text{Log}$ has no minimum element, it has a maximum and a coatom (that under Vopěnka’s Principle is unique).
Term-equivalence and compatible expansions

**Definition**
Let $\vdash$ and $\vdash'$ be two logics.

1. $\vdash$ and $\vdash'$ are term-equivalent if there are translations $\tau$ of $\vdash$ into $\vdash'$ and $\rho$ in the other direction such that
   $\langle A, F \rangle = \langle A^\tau, F \rangle$ and $\langle B, G \rangle = \langle B^\rho, G \rangle$
   for every $\langle A, F \rangle \in Mod^{Su}(\vdash')$ and $\langle B, G \rangle \in Mod^{Su}(\vdash)$.

2. $\vdash'$ is a compatible expansion of $\vdash$ if the identity is a translation of $\vdash$ into $\vdash'$.

**Remark**
$\vdash \leq \vdash'$ iff $\vdash'$ is term-equivalent to a compatible expansion of $\vdash$.

Taylorian products of algebras

**Definition**
Let $\{L_i : i \in I\}$ be a set algebraic languages.
We define a new language $\bigotimes_{i \in I} L_i$ by considering as $n$-ary operations symbols the sequences
$\langle t_i(x_1, \ldots, x_n) : i \in I \rangle$
where $t_i$ is an $n$-ary term of $L_i$ in the variables $x_1, \ldots, x_n$.

**Definition**
Let $\{A_i : i \in I\}$ be a set of algebras respectively of language $L_i$.
The Taylorian product of this family is the algebra $\bigotimes_{i \in I} A_i$ of type
$\bigotimes_{i \in I} L_i$ with universe $\prod_{i \in I} A_i$ and operations defined as
$\langle t_i(i) : i \in I \rangle(a_1, \ldots, a_n) = \langle t_i^A((a_1(i), \ldots, a_n(i)) : i \in I)\rangle$.

**Remark**
Observe that Taylorian products of huge families of logics are formulated in huge sets of variables.

**Corollary**
$\text{Log}$ has infima of families indexed by sets. More precisely,
$\bigwedge_{i \in I} \vdash_i = \bigwedge_{i \in I} \bigwedge_{i \in I} \vdash_i$.
Leibniz conditions and Leibniz classes

Definition
1. A strong Leibniz condition $\Phi$ is a logic $\vdash \Phi$.
2. A logic $\vdash$ satisfies $\Phi$ if $\vdash \Phi \leq \vdash$.
3. A Leibniz condition $\Phi$ is a sequence of logics $\Psi := \{\vdash \alpha: \alpha \in \text{ORD}\}$ such that if $\alpha \leq \beta$, then $\vdash \beta \leq \vdash \alpha$.
4. A logic $\vdash$ satisfies $\Psi$ if $\vdash \alpha \leq \vdash$ for some $\alpha \in \text{ORD}$.
5. Mod($\Psi$) is the class of logics satisfying $\Psi$.
6. A class of logics $K$ is a (strong) Leibniz class if $K = \text{Mod}(\Psi)$ for some (strong) Leibniz condition $\Psi$.

Equivalentiality is a Leibniz condition

Example
- For every $\alpha \in \text{ORD}$, consider the language $\{-\circ_\beta: \beta \leq \alpha\}$.
- Define $\Delta_\alpha(x, y) := \{x -\circ_\beta y: \beta \leq \alpha\}$.
- Let $\vdash_\alpha$ be the logic defined by the rules
  $\emptyset \vdash \Delta_\alpha(x, x)$
  $x, \Delta_\alpha(x, y) \vdash y$
  $\Delta_\alpha(x_1, y_1) \cup \Delta_\alpha(x_2, y_2) \vdash \Delta_\alpha(x_1 -\circ_\beta x_2, y_1 -\circ_\beta y_2)$.
- Consider the Leibniz condition $\Psi := \{\vdash_\alpha: \alpha \in \text{ORD}\}$.
- Mod($\Psi$) is the class of equivalential logics with theorems.

Semantic description of Leibniz classes

- Leibniz classes can be characterized as follows:

Theorem
Let $K$ be a class of logics. TFAE:
1. $K$ is a Leibniz class.
2. $K$ is closed under term-equivalence, compatible expansions and Taylorian products indexed by sets.
3. There is a complete filter $F$ of $\text{Log}$ such that $K = \{\vdash: [\vdash] \in F\}$.

- In this picture,
  Strong Leibniz classes = principal filters of $\text{Log}$.

The Leibniz hierarchy revisited

- We propose to adopt the following:

Convention
Leibniz hierarchy = poset of Leibniz classes of logics.

Some motivations:
- This perspective subsumes Maltsev conditions.
- Leibniz classes captures the interaction between syntactic conditions and the behaviour of the Leibniz operator.
- Leibniz classes are not too general. They do not include metalogical properties and the Frege hierarchy.
Indecomposable Leibniz classes

- This order-theoretic perspective allows to single out the fundamental bricks of the Leibniz hierarchy:

**Definition**
A Leibniz class $K$ is *indecomposable* if it is meet-irreducible among Leibniz classes.

- The class of logics with theorems is indecomposable.

**Hopeless Lemma**
Let $K$ be a Leibniz class such that:
1. The members of $K$ have theorems.
2. There is a logic with theorems outside $K$.
Then $K$ is decomposable.

- Almost all reasonable Leibniz classes are decomposable.

Indecomposability among logics with theorems

- We use the following principle independent from GNB.

**Vopěnka’s Principle**
Every prevariety is a generalized quasi-variety.

**Theorem**
Under Vopěnka’s Principle, the classes of truth equational and assertional logics are *indecomposable* among logics with theorems.

**Theorem**
The classes of order-algebraizable and equivalential logics with theorems are *decomposable* among logics with theorems.

- It is *open* whether the class of protoalgebraic logics is decomposable among logics with theorems.
Finitely presentable deductive systems

**Definition**
1. A logic is **finitely presentable** if it is finitary, axiomatizable by a finite set of finite rules, and formulated in a finite language.
2. A **finitely presentable Leibniz condition** is a sequence of finitely presentable and finitely equivalential logics \( \Psi = \{\vdash_n : n \in \omega\} \) such that if \( n \leq m \), then \( \vdash_m \leq \vdash_n \).
3. A class of logics \( K \) is a **finitely presentable Leibniz class** if \( K = \text{Mod}(\Psi) \) for some finitely presentable Leibniz condition \( \Psi \).

**Convention**
finite companion of the Leibniz hierarchy = poset of finitely presentable Leibniz classes.

The Maltsev hierarchy

**Definition**
1. A **Maltsev condition** is a sequence of finitely presentable varieties \( \Psi = \{V_n : n \in \omega\} \) such that if \( n \leq m \), then \( V_m \leq V_n \).
2. A class of varieties \( K \) is a **Maltsev class** if \( K = \text{Mod}(\Psi) \) for some Maltsev condition \( \Psi \).

**Theorem**
A class of varieties \( K \) is a Maltsev class iff there is a finitely presentable Leibniz class \( M \) of 2-deductive systems such that \( K = \{V : V \text{ is a variety and } \vdash V \in M\} \).

Future directions

- Can we have a suitable version of Taylor terms for logic?
- Can we prove that non-trivial Leibniz conditions implies the validity of some non-trivial (quasi)-equation involving the Leibniz operator?
- Are protoalgebraic logics indecomposable/prime?

Finally...

...thank you for coming!