The computational complexity of the Leibniz hierarchy

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Logics

Definition
A logic \( \vdash \) is a consequence relation over the set of formulas \( Fm \) of an algebraic language, which is substitution invariant in the sense that

\[
\text{if } \Gamma \vdash \varphi, \text{ then } \sigma(\Gamma) \vdash \sigma(\varphi)
\]

for all substitutions \( \sigma : Fm \to Fm \).

▶ Logics are consequence relations (as opposed to sets of valid formulas).
▶ Example: IPC is the logic defined as follows:

\( \Gamma \vdash_{\text{IPC}} \varphi \iff \) for every Heyting algebra \( A \) and \( \vec{a} \in A \),

\[
\text{if } \Gamma^A(\vec{a}) = 1, \text{ then } \varphi^A(\vec{a}) = 1.
\]

Relative equational consequence

Definition
Let \( K \) be a class of similar algebras. Given a set of equations \( \Theta \cup \{ \varphi \approx \psi \} \), we define

\[
\Theta \models_K \varphi \approx \psi \iff \text{for every } A \in K \text{ and } \vec{a} \in A,
\]

\[
\text{if } \epsilon^A(\vec{a}) = \delta^A(\vec{a}) \text{ for all } \epsilon \approx \delta \in \Theta,
\]

\[
\text{then } \varphi^A(\vec{a}) = \psi^A(\vec{a}).
\]

The relation \( \models_K \) is the equational consequence relative to \( K \).

▶ Example: If \( K \) is the variety of Heyting algebras, then

\[
\varphi \approx 1, \varphi \to \psi \approx 1 \models_K \psi \approx 1.
\]

Algebraizable logics

Example: Consider

\( \text{IPC} = \) intuitionistic propositional logic
\( \text{HA} = \) variety of Heyting algebras

▶ Pick the translations between formulas and equations:

\[
\varphi \mapsto \varphi \approx 1
\]

\[
\alpha \approx \beta \mapsto \{ \alpha \leftrightarrow \beta \}.
\]

▶ These translations allow to equi-interpret \( \vdash_{\text{IPC}} \) and \( \models_{\text{HA}} \):

\[
\Gamma \vdash_{\text{IPC}} \varphi \iff \{ \gamma \approx 1 : \gamma \in \Gamma \} \models_{\text{HA}} \varphi \approx 1
\]

\[
\Theta \models_{\text{HA}} \varphi \approx \psi \iff \{ \alpha \leftrightarrow \beta : \alpha \approx \beta \in \Theta \} \vdash_{\text{IPC}} \{ \varphi \leftrightarrow \psi \}.
\]

▶ Moreover, the translations are one inverse to the other:

\[
\varphi \approx \psi \models_{\text{HA}} \varphi \leftrightarrow \psi \approx 1 \text{ and } \varphi \vdash_{\text{IPC}} \varphi \leftrightarrow \psi.
\]

▶ Hence \( \vdash_{\text{IPC}} \) and \( \models_{\text{HA}} \) are essentially the same.
Intuitive idea: a logic $\vdash$ is algebraizable when it can be essentially identified with a relative equational consequence $\vDash_K$.

**Definition**

A logic $\vdash$ is **algebraizable** when there exists:

1. A class of algebras $K$ (of the same type as $\vdash$);
2. A set of equations $\tau(x)$ in one variable $x$;
3. A set of formulas $\rho(x, y)$ in two variables $x$ and $y$ such that $\tau$ and $\rho$ equi-interpret $\vdash$ and $\vDash_K$:

\[
\Gamma \vdash \varphi \iff \tau(\Gamma) \vDash_K \tau(\varphi) \\
\Theta \vDash_K \varphi \approx \psi \iff \rho(\Theta) \vdash \rho(\varphi, \psi)
\]

and the two interpretations are one inverse to the other:

\[
\varphi \approx \psi \iff \models_K \tau(a, b) \subseteq F \\
a \in F \iff A \models \tau(a).
\]

Since finitely generated free algebras over $\forall(A)$ are finite, we can just check the existence of the sets $\rho(x, y)$ and $\tau(x)$.

Hence the Semantic Algebraization Problem is in **EXPTIME**.

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**Algebraization Problem**

- We study the computational aspects of the following problem:

**Algebraization Problem**

Given a logic $\vdash$, determine whether $\vdash$ is algebraizable or not.

- Logic can be presented (at least) in two ways:
  
  **syntactically** = by means of Hilbert calculi
  
  **semantically** = by means of collections of logical matrices.

**Theorem (M. 2015)**

The Algebraization Problem for logics presented by finite consistent Hilbert calculi is **undecidable**.

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**A useful **EXPTIME**-complete problem**

- We want to prove that the Semantic Algebraization Problem is complete for **EXPTIME**.
- We need to construct a polynomial-time reduction to such a complete problem.

**The Problem Gen-Clo**

Given a finite algebra $A$ of finite type and a function $h: A^n \rightarrow A$, determine whether $h$ belongs to the clone of $A$ or not.

- Gen-Clo$_1$ is the same problem, restricted to the case where $h$ is unary and the operations of $A$ are at most ternary.

**Theorem (Bergman, Juedes, and Slutzki)**

Both Gen-Clo and Gen-Clo$_1$ are complete for **EXPTIME**.

- We will construct a polynomial reduction of Gen-Clo$_1$ to the Semantic Algebraization Problem.
Reduction

Pick an input \( \langle A, h \rangle \) for \( \text{Gen-Clo}_3^1 \). We define a new algebra \( A^\flat \) as:

- The universe of \( A^\flat \) is eight disjoint copies \( A_1, \ldots, A_8 \) of \( A \):
  
  \[ \{a_1^{m_1}, \ldots, a_n^{m_n}\} \]
  
  for some \( a_1, \ldots, a_n \in A \) and \( m_1, \ldots, m_n \leq 8 \).
- The basic operation of \( A^\flat \) are as follows:
  1. For every \( n \)-ary basic \( f \) of \( A \), we add an operation \( \hat{f} \) on \( A^\flat \) as
     \[ \hat{f}(a_1^{m_1}, \ldots, a_n^{m_n}) := f^A(a_1, \ldots, a_n)^5. \]
  2. Then we add to \( A^\flat \) the following operation \( \Box \):
     \[ \Box(a^m) := \begin{cases} 
     a^m & \text{if } m = 1 \text{ or } m = 2 \\
     a^{m-1} & \text{if } m \text{ is even and } m \geq 3 \\
     a^{m+1} & \text{if } m \text{ is odd and } m \geq 3.
     \end{cases} \]

3. Finally we add to \( A^\flat \) the following operation \( \heartsuit \):
   \[ \heartsuit(a^m, b^n, c^k) := \begin{cases} 
     a^1 & \text{if } a^m = c^k \text{ and } h(a)^5 = b^n \\
     a^2 & \text{if } a^m = c^k \text{ and } h(a)^5 = b^n \text{ and } m \in \{1, 3, 4\} \\
     a^4 & \text{if } m, k \in \{1, 3, 4\} \text{ and } (\text{either } a^m \neq c^k \text{ or } h(a)^5 \neq b^n) \\
     a^7 & \text{if } \{m, k\} \cap \{2, 5, 6, 7, 8\} \neq \emptyset \text{ and } (\text{either } a^m \neq c^k \text{ or } h(a)^5 \neq b^n). 
     \end{cases} \]

- Then define \( F \subseteq A^\flat \) as follows: \( F := A_1 \cup A_2 \).
- The pair \( \langle A^\flat, F \rangle \) is a finite reduced matrix of finite type, and thus an input for the Semantic Algebraization Problem!

Remark

Since the arity of the operations of \( A \) is bounded by 3, the matrix \( \langle A^\flat, F \rangle \) can be constructed in polynomial time.

Hardness result

There is a polynomial-time reduction of Gen-Clo_3^1 to the Semantic Algebraization Problem, i.e. given a finite algebra \( A \) of finite type, whose basic operations are at most ternary, and a unary map \( h: A \rightarrow A \), TFAE:

1. \( h \) belongs to the clone of \( A \).
2. The logic induced by the matrix \( \langle A^\flat, F \rangle \) is algebraizable.

Corollary

The Semantic Algebraization Problem is complete for EXPTIME.
Further questions

- A similar situation appears in the study of Mal'cev conditions:

**Theorem (Freese and Valeriote)**

The problem of determining whether a finite algebra $A$ of finite type generates a congruence distributive (resp. modular) variety is complete for $\text{EXPTIME}$.

- However, the above problems become tractable when $A$ is idempotent, i.e. when for every operation $f$ of $A$ and $a \in A$:

  $$f^A(a, \ldots, a) = a$$

**Open Problem**

Find tractability conditions for Semantic Algebraization Problem.

- **Remark**: idempotency will not work here, since no idempotent non-trivial matrix determines an algebraizable logic.

Finally...

...thank you for coming!