Semi-Constructive versions of the Rasiowa-Sikorski Lemma and Possibility Semantics for Intuitionistic Logic

Guillaume Massas

University of California, Irvine

TACL 2017, June 29

Plan

Motivation

- The following are well-known and important results about Boolean algebras and classical predicate logic (CPL):
  - The Rasiowa-Sikorski Lemma for Boolean algebras
  - Stone's representation theorem
  - The completeness of CPL with respect to Tarskian semantics
- Over time, these results have been generalized in two different ways:
  - By moving away from *Boolean algebras*, and extending the results to distributive lattices, Heyting algebras and intuitionistic logic (mathematical program);
  - By moving away from *classical mathematics*, in particular working under *fragments* of the axiom of choice instead of the full *AC* (metamathematical program).

Motivation

- My goal is to combine the two programs, and provide generalizations of those classical results to DL and HA by using only fragments of AC.

	DL and HA	BA
Fragments of AC	?	
Full AC		

Motivation

- My goal is to combine the two programs, and provide generalizations of those classical results to DL and HA by using only fragments of AC.

	DL and HA	BA
Fragments of AC	?	
Full AC		

- In all three cases, the same idea will appear, namely that we have to work with pairs of filters and ideals rather than just with filters.

Outline

Plan

The Rasiowa-Sikorski Lemma

Lemma (Rasiowa-Sikorski)

Let B be a Boolean algebra and Q a countable set of subsets of B with meets existing in B. Then for any $a \in B$, if $a \neq 0$, then there is an ultrafilter p over B such that:

- $a \in p$;
- For any $X \in Q$, if $X \subseteq p$, then $\bigwedge X \in p$.

The Rasiowa-Sikorski Lemma

- Rasiowa and Sikorski's original proof was an application of the Baire Category Theorem for compact Hausdorff spaces (BCT) to the dual Stone space of a Boolean algebra.
- Rauszer-Sabalski(1975), Görnemann(1971), and more recently Goldblatt(2012) showed how to generalize this result to DL and HA.
- These proofs however are non-constructive, because they rely on the Boolean Prime Ideal Theorem (BPI) or the Prime Filter Theorem (PFT).

Non-Constructive Principles

- On the other hand, (Goldblatt 1985) remarks that the Rasiowa-Sikorski Lemma is equivalent to the conjunction of BPI and Tarski's Lemma:

Tarski's Lemma

Let B be a Boolean algebra and Q a countable set of subsets with meets existing in B. Then for any $a \in B$, if $a \neq 0$, then there exists a filter F over B such that:

- $a \in F$;
- for any $X \in Q$, either $\bigwedge X \in F$, or $\neg x \in F$ for some $x \in X$.

Non-Constructive Principles

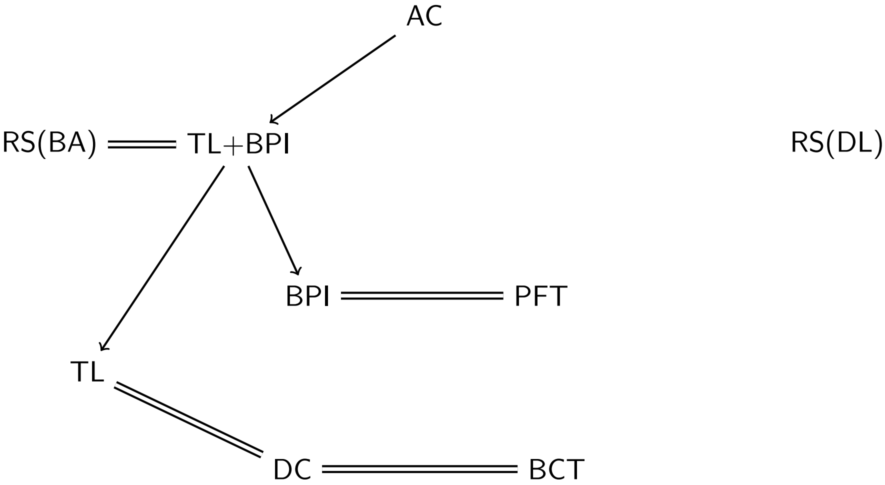
- On the other hand, (Goldblatt 1985) remarks that the Rasiowa-Sikorski Lemma is equivalent to the conjunction of BPI and Tarski's Lemma:

Tarski's Lemma

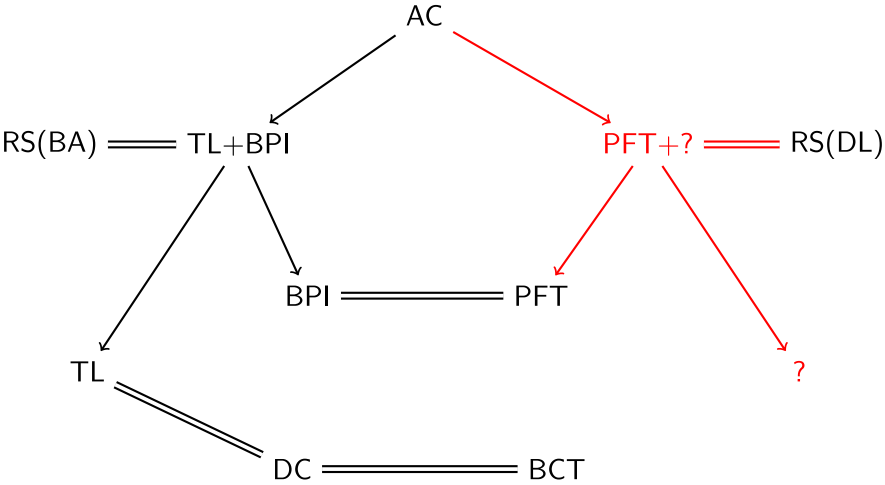
Let B be a Boolean algebra and Q a countable set of subsets with meets existing in B. Then for any $a \in B$, if $a \neq 0$, then there exists a filter F over B such that:

- $a \in F$;
- for any $X \in Q$, either $\bigwedge X \in F$, or $\neg x \in F$ for some $x \in X$.

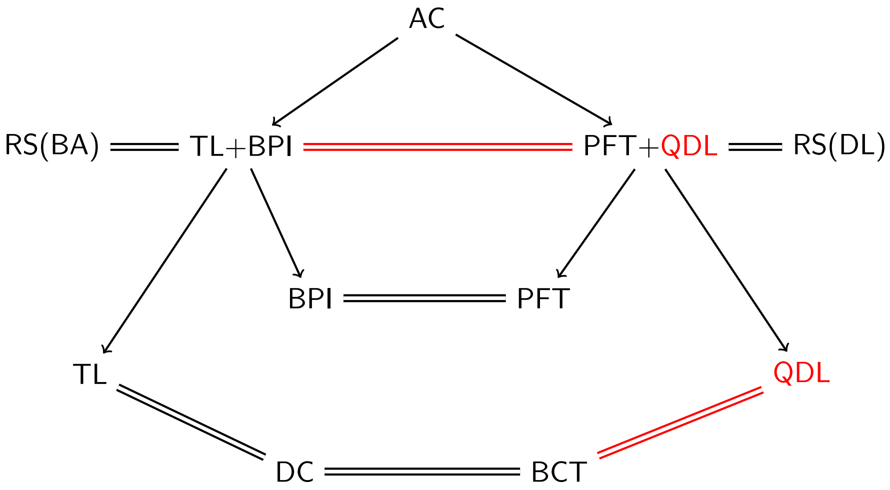
- Goldblatt also proves that Tarski's Lemma is equivalent over ZF to several other statements, including BCT and the Axiom of Dependent Choices (DC).


AC
RS(BA)
TL+BPI
RS(DL)
BPI
PFT
TL
DC
BCT

Non-Constructive Principles



Non-Constructive Principles



The Q-Lemma for Distributive Lattices

Q-Lemma for DL

Let L be a distributive lattice and Q_M and Q_J two countable sets of distributive meets and joins existing in L respectively. Then for any $a, b \in L$ such that $a \nleq b$, there exists a pair (F, I) over L such that:

i) $a \in F$ and $b \in I$, $F \cap I = \emptyset$;

ii) For any $\bigwedge X \in Q_M$, either $\bigwedge X \in F$, or there exists $x \in X \cap I$;

iii) For any $\bigvee Y \in Q_J$, either $\bigvee Y \in I$, or there exists $y \in Y \cap F$.

Proof.

Recall first that L is distributive iff for any $a, b, c \in L$, if $a \leq b \vee c$ and $a \wedge c \leq b$ then $a \leq b$. Hence for any $X \in Q_M$, $Y \in Q_J$, $a, b \in L$, by distributivity if $a \nleq b$, then either $a \nleq x \vee b$ for some $x \in X$, or $a \wedge \bigwedge X \nleq b$, and dually either $a \wedge y \nleq b$ for some $y \in Y$, or $a \nleq \bigvee Y \vee b$. Order all subsets in Q_M and all subsets in Q_J, and...

The Q-Lemma for Distributive Lattices

Q-Lemma for DL

Let L be a distributive lattice and Q_M and Q_J two countable sets of subsets with distributive meets and joins existing in L respectively. Then for any $a, b \in L$ such that $a \nleq b$, there exists a pair (F, I) over L such that:

i) $a \in F$ and $b \in I$, $F \cap I = \emptyset$;

ii) For any $\bigwedge X \in Q_M$, either $\bigwedge X \in F$, or there exists $x \in X \cap I$;

iii) For any $\bigvee Y \in Q_J$, either $\bigvee Y \in I$, or there exists $Y \in Y \cap F$.

Proof.

...construct a descending sequence $\{a_n\}_{n\in\omega}$ and an increasing sequence $\{b_n\}_{n\in\omega}$ such that $a_0 = a$, $b_0 = b$, $a_i \nleq b_i$ for all $i \in \omega$, and for $X_i \in Q_M$, either $a_{2i} \leq \bigwedge X_i$ or $b_{2i} \geq x$ for some $x \in X_i$, and for $Y_j \in Q_J$, either $b_{2i+1} \geq \bigvee Y_j$ or $a_{2j+1} \leq y$ for some $y \in Y_j$. The upward and downward closure of $\{a_n\}_{n\in\omega}$ and $\{b_n\}_{n\in\omega}$ respectively yield the required pair. □

The Q-Lemma for Heyting Algebras

Q-Lemma for HA

Let A be a Heyting algebra, and Q_M and Q_J two countable $(\bigwedge, \to)$-complete sets of distributive meets and joins existing in A respectively. Then for any Q-pair (F, I) over A and any $a, b \in A$, if $a \to b \notin F$, then there exists a Q-pair (F', I') such that $F \cup \{a\} \subseteq F'$ and $b \in I'$.

Proof.

This is an "internalized" version of the previous one. □

Plan

Non-constructive representation theorems

- It is well-known that every Boolean algebra can be represented as a subalgebra of the powerset of its dual Stone space. (Stone representation theorem).
- This result generalizes to distributive lattices and Heyting algebras: every distributive lattice can be represented as a subalgebra of the upsets of its dual Priestley (resp. Esakia) space. (Priestley (resp. Esakia) representation theorem).
- Those results are non-constructive: they provide prime-filter based representations and therefore rely on the Prime Filter Theorem.

Constructive representation for Boolean algebras

- There exists however an elegant choice-free representation theorem for Boolean algebras: the filter space construction.

Definition (Filter-space)

Let B be a Boolean algebra. The filter-space of B is the topological space (S_B, τ), where S_B is the set of all proper filters over B, and τ is the upset topology induced by the inclusion ordering.

Lemma

Let B be a Boolean algebra and (S_B, τ) its filter space. Then the Stone map $|\cdot| : B \to \mathscr{P}(S_B)$ is an embedding of B into the regular opens $\mathrm{RO}(S_B)$ of (S_B, τ).

- In fact, $\mathrm{RO}(S_B)$ is the canonical extension of B.

Constructive representation for Boolean algebras

- This result relies on the well-known topological fact that the regular opens of any topological space form a complete Boolean algebra.
- In point-free topological terms: the *IC* operator (*Interior-Closure*) is the double negation nucleus on the frame of opens of any topological space.
- Can we follow a similar strategy for all distributive lattices?

Refined bi-topological spaces

Definition

A *refined bi-topological space* is a bi-topological space (X, τ_1, τ_2) such that $\tau_1 \subseteq \tau_2$

Lemma

Let (X, τ_1, τ_2) be a refined bi-topological space. Then the operator $I_1 C_2$ (Interior in τ_1, Closure in τ_2) is a nucleus on the frame of opens in τ_1.

Corollary

Let (X, τ_1, τ_2) be a refined bi-topological space. Then $\mathrm{RO}_{12}(X)$ is a *cHA*.

Constructive representation theorem for distributive lattices

Definition

Let L be a lattice. A *pseudo-complete pair* over L is a pair (F, I) such that:

- F is a filter, I is an ideal, and $F \cap I = \emptyset$ (compatible pair);
- For any $a \in F$, $b \in I$ and $c \in L$, if $a \wedge c \leq b$, then $c \in I$ (Right Meet Property);
- For any $a \in F$, $b \in I$ and $c \in L$, if $a \leq b \vee c$, then $c \in F$ (Left Join Property).

Constructive representation theorem for distributive lattices

Definition

Let L be a lattice. A *pseudo-complete pair* over L is a pair (F, I) such that:

- F is a filter, I is an ideal, and $F \cap I = \emptyset$ (compatible pair);
- For any $a \in F$, $b \in I$ and $c \in L$, if $a \wedge c \leq b$, then $c \in I$ (Right Meet Property);
- For any $a \in F$, $b \in I$ and $c \in L$, if $a \leq b \vee c$, then $c \in F$ (Left Join Property).

Lemma (ZF)

Let L be a lattice. Then L is distributive iff for any compatible pair (F, I) over L, there exists a pseudo-complete pair (F^*, I^*) such that $F \subseteq F^*$ and $I \subseteq I^*$.

Constructive representation theorem for distributive lattices

Definition

Let L be a distributive lattice. The *canonical filter-ideal space* is the refined bitopological space (S_L, τ_1, τ_2), where S_L is the set of all pseudo-complete pairs over L, and τ_1 and τ_2 are the upset topologies induced by the filter inclusion ordering and the filter-ideal inclusion ordering respectively.

Theorem

Let L be a distributive lattice, and (S_L, τ_1, τ_2) its canonical filter-ideal space. Then the Stone map: $|\cdot| : L \to \mathscr{P}(S_L)$ defined by $|a| = \{(F, I) \in S_L \ ; \ a \in F\}$ is a DL-embedding of L into $\mathrm{RO}_{12}(S_L)$. Additionally, if L is a Heyting algebra, then $|\cdot|$ is a HA-embedding.

A note on completions

- For any distributive lattice L with canonical filter-ideal space (S_L, τ_1, τ_2), $\mathrm{RO}_{12}(S_L)$ is the canonical extension of L.

- But one can also slightly modify the canonical filter-ideal space (S_L, τ_1, τ_2) of a distributive lattice L in order to realize various kind of completions as $\mathrm{RO}_{12}(S_L)$.

- For example, letting τ_+ and τ_- be the topologies generated by the bases $\{|a| \ ; \ a \in L\}$ and $\{|a|^- \ ; \ a \in L\}$ respectively, we have that $\mathrm{RO}_{+-}(S_L)$ is the MacNeille completion of L.

- Alternatively, for Q_M and Q_J as above, letting Q_L be the set of all pseudo-complete Q-pairs, we have that $\mathrm{RO}_{12}(Q_L)$ is a completion of L that preserves precisely all infinite meets in Q_M and all infinite joins in Q_J. The proof requires the Q-Lemma.

Plan

Completeness results

- Rasiowa and Sikorski applied their lemma to the Lindenbaum-Tarski algebra of *CPL* and gave a new proof of the completeness of CPL with respect to *Tarskian* models.
- Similarly, a combination of the Rasiowa-Sikorski Lemma for Heyting algebras and Esakia representation theorem yields a very similar proof of the Kripke completeness of Intuitionistic Predicate Logic with Constant Domains (IPL).
- But *semi-constructive* methods can also be used to prove the completeness of CPL with respect to *possibility semantics*.

Possibility semantics

Definition (Regular map)

Let $(P, \leq_1,)$ and $(Q, \leq_2)$ be posets. A map $f : P \to Q$ is *regular* if for every $x \in P$ and $a \in Q$, if for all $y \geq_1 x$ there is $z \geq_1 y$ such that $a \leq_2 f(z)$, then $a \leq_2 f(x)$.

Definition (First-order Possibility model)

A (first-order) *possibility model* is a tuple $(X, \leq, D, h, I)$ such that $\leq$ is a partial order on X, D is a domain of individuals, h is an assignment from $Var(IPL)$ to D and for each $R^n \in Rel(IPL)$, $I(R^n)$ is a monotone and regular map from X to $\mathscr{P}(D^n)$.

Possibility semantics

Definition (Valuation)

Let $(X, \leq, D, h, I)$ be a first-order possibility model. The valuation $I^* : Fm(IPL) \to \mathrm{RO}(X)$ is defined inductively as follows:

- $s \vDash \top$ always, $s \vDash \bot$ never;
- $s \vDash R^n(v_1, ..., v_n)$ iff $(h(v_1), ..., h(v_n)) \in I(R^n)(x)$ for any $R^n \in Rel(IPC)$, $v_1, ..., v_n \in Var(IPL)$;
- $s \vDash \phi \wedge \psi$ iff $s \vDash \phi$ and $s \vDash \psi$;
- $s \vDash \phi \vee \psi$ iff for all $y \geq s$ there is $z \geq y$ such that $z \vDash \phi$ or $z \vDash \psi$;
- $s \vDash \phi \to \psi$ iff for all $y \geq s$, if $y \vDash \phi$, then $y \vDash \psi$;
- $s \vDash \forall x \phi(x)$ iff $s \vDash \phi(x)[a/x]$ for all $a \in D$;
- $s \vDash \exists x \phi(x)$ iff for all $y \geq s$ there is $z \geq y$ such that $z \vDash \phi(x)[a/x]$ for some $a \in D$.

A formula ϕ is valid on a possibility model $(X, \leq, D, h, I)$ if $I^*(\phi) = X$.

Possibility semantics

- Classical Predicate Logic (CPL) is sound and complete with respect to first-order possibility models.
- The completeness proof involves the construction of a term model based on the filter space of the Lindenbaum-Tarski algebra of CPL, restricted to the set of Q-filters, for $Q = \{\{\phi(x) \; ; \; x \in Var\} \; ; \; \phi \in Fml\}$, and uses Tarski's Lemma.

Intuitionistic Possibility semantics

Definition (Refined regular map)

Let $(P, \preccurlyeq_1, \preccurlyeq_2)$ be a refined preorder and $(Q, \leq)$ a poset. A map $f : P \to Q$ is *refined regular* if for every $x \in P$ and $a \in Q$, if for all $y \succcurlyeq_1 x$ there is $z \succcurlyeq_2 y$ such that $a \leq f(z)$, then $a \leq f(x)$.

Definition (First-order Possibility model)

A (first-order) *intuitionistic possibility model* (IP-model) is a tuple $(X, \preccurlyeq_1, \leq_2, D, h, I)$ such that $(X, \preccurlyeq_1, \leq_2)$ is a refined bi-preorder, $\leq_2$ is a partial order on X, D is a domain of individuals, h is an assignment from $Var(IPL)$ to D and for each $R^n \in Rel(IPL)$, $I(R^n)$ is a monotone and refined regular map from X to $\mathscr{P}(D^n)$.

IP semantics

Definition (Valuation)

Let $(X, \preccurlyeq_1, \leq_2, D, h, I)$ be a first-order IP model. The valuation $I^* : Fm(IPL) \to \mathrm{RO}_{12}(X)$ is defined inductively as follows:

- $s \Vdash \top$ always, $s \Vdash \bot$ never;
- $s \Vdash R^n(v_1, ..., v_n)$ iff $(h(v_1), ..., h(v_n)) \in I(R^n)(x)$ for any $R^n \in Rel(IPC)$, $v_1, ..., v_n \in Var(IPL)$;
- $s \Vdash \phi \wedge \psi$ iff $s \Vdash \phi$ and $s \Vdash \psi$;
- $s \Vdash \phi \vee \psi$ iff for all $y \succcurlyeq_1 s$ there is $z \geq_2 y$ such that $z \Vdash \phi$ or $z \Vdash \psi$;
- $s \Vdash \phi \to \psi$ iff for all $y \geq s$, if $y \Vdash \phi$, then $y \Vdash \psi$;
- $s \Vdash \forall x \phi(x)$ iff $s \Vdash \phi(x)[a/x]$ for all $a \in D$;
- $s \Vdash \exists x \phi(x)$ iff for all $y \succcurlyeq_1 s$ there is $z \geq_2 y$ such that $z \Vdash \phi(x)[a/x]$ for some $a \in D$.

A formula ϕ is valid on an IP model $(X, \preccurlyeq_1, \leq_2, D, h, I)$ if $I^*(\phi) = X$.

IP semantics

- "Intuitive" picture of an IP-model $(X, \preccurlyeq_1, \leq_2, D, h, I)$: X represents a set of (partial) states of information.
- Two agents, *Eloise* and *Abelard*, order these states of information as *possible developments* of one another.
- Namely, for any $x, y \in X$:
  - $x \preccurlyeq_1 y$ iff y is a possible development of x according to Eloise;
  - $x \leq_2 y$ iff y is a possible development of x according to Abelard.

IP semantics

- "Intuitive" picture of an IP-model $(X, \preccurlyeq_1, \leq_2, D, h, I)$: X represents a set of (partial) states of information.
- Two agents, *Eloise* and *Abelard*, order these states of information as *possible developments* of one another.
- Namely, for any $x, y \in X$:
  - $x \preccurlyeq_1 y$ iff y is a possible development of x according to Eloise;
  - $x \leq_2 y$ iff y is a possible development of x according to Abelard.
- Eloise and Abelard are in an asymmetric Student / Instructor relation: Abelard knows at least as much as Eloise (i.e. $\leq_2 \subseteq \preccurlyeq_1$).
- In particular, Eloise may fail to distinguish two different states of information, while Abelard doesn't. (i.e. $\preccurlyeq_1$ is a preorder vs. $\leq_2$ is a partial order)

IP semantics

- Idea behind the forcing relation: Abelard is testing Eloise's knowledge of some formula at a given state of information.
- Eloise knows ϕ iff for every question asked by Abelard, Eloise can reply ϕ in a way that is satisfactory to Abelard.
- Formally: $s \Vdash \phi$ iff $\forall y \succcurlyeq_1 s \, \exists z \geq_2 y : z \Vdash \phi$.

IP semantics

- Intuitionistic Predicate Logic with Constant Domains (IPL) is sound and complete with respect to first-order possibility models.
- The completeness proof involves the construction of a term model based on the canonical filter-ideal space of the Lindenbaum-Tarski algebra of IPL, restricted to the set of Q-filters, for $Q_M = Q_J = \{\{\phi(x)\ ;\ x \in Var\}\ ;\ \phi \in Fml\}$, and uses the Q-Lemma for DL and HA.

Kripke and possibility frames as degenerate IP-frames

Lemma

- *Let $M_1 := (X, \leq_1)$ be a Kripke model. Then $M_2 := (X, \leq_1, \Delta_X)$, where Δ_X is the identity on X, is an IP-model. Moreover, for any formula $\phi \in Fm_{IPC}$ and any $x \in X$, $M_1, x \Vdash \phi$ iff $M_2, x \Vdash \phi$.*
- *Let $M_1 := (X, \leq_1)$ be a possibility model. Then $M_2 := (X, \leq_1, \leq_1)$, is an IP-model. Moreover, for any formula $\phi \in Fm_{IPC}$ and any $x \in X$, $M_1, x \Vdash \phi$ iff $M_2, x \Vdash \phi$.*

- Intuitively: Kripke frames are those IP-frames in which Abelard knows much more than Eloise. Possibility frames are those IP-frames in which Eloise knows as much as Abelard.

The semantic hierarchy for intuitionistic logic

- The following hierarchy of semantics for IPC is well-known:

$$\text{Kripke frames} \prec \text{Topological spaces} \prec \text{Heyting algebras}$$

Lemma

For any complete Heyting algebra A, there exists an IP-frame $(X, \preccurlyeq_1, \leq_2)$ such that A is isomorphic to $\mathrm{RO}_{12}(X)$.

- This means that we can complete the hierarchy as follows:

$$\text{Kripke frames} \prec \text{Topological spaces} \prec \text{IP-frames} \prec \text{Heyting algebras}$$

- In fact, the propositional fragment of IP-semantics is equivalent to Dragalin semantics (Bezhanishvili and Holliday 2016), and a restriction of FM-frames for lax logic (Fairtlough-Mendler 1997).

Plan

Open problems

- Can we generalize the Q-Lemma to varieties of non-distributive lattices?
- What kind of completions of DL and HA are realized as refined regular opens of filter-ideal spaces with bitopologies in the interval $[(\tau_+, \tau_-), (\tau_1, \tau_2)]$?
- Weakening of Kuznetsov's problem: Is every intermediate logic complete with respect to some class of IP-frames?

Thank You!