Injectivity of ordered and naturally ordered projection algebras

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Overview

Abstract

A **projection algebra** is a set with an action of the monoid of extended natural numbers with the minimum as the binary operation.
In this article,
• We consider **injectivity** of **ordered projection algebras**, that is partially ordered projection algebras whose action is monotone.
• We characterize cyclic injective ones as complete posets.
• We show that injectivity of arbitrary ordered projection algebras is in some sense related to injectivity of naturally ordered projection algebras.
• We give some Baer criteria by studying some kinds of weak injectivity for (naturally) ordered projection algebras such as ideal injectivity, $\mathbb{N}$-injectivity, and regular injectivity and study the relations between them.

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Action of a monoid on a set: M-set

Definition

Let M be a monoid. A (*left*) *M-set* is a set A equipped with an action $\lambda : M \times A \to A$, $(s, a) \rightsquigarrow sa$, such that have $1a = a$ and $(st)a = s(ta)$, for all $a \in A$, and $s, t \in M$.

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- An element θ of an M-set is called a *zero* or a *fixed* element if $s\theta = \theta$ for all $s \in M$.
- The set of all fixed elements of A is denoted by $Fix(A)$.

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- An element θ of an M-set is called a *zero* or a *fixed* element if $s\theta = \theta$ for all $s \in M$.
- The set of all fixed elements of A is denoted by $Fix(A)$.
- A map $f : A \to B$ between M-sets is called *action preserving* if $f(sa) = sf(a)$, for all $a \in A$, $s \in M$.

Projection Algebra

- M-sets for the monoid $M = (\mathbb{N}^\infty, \min)$, where $\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$, $\mathbb{N}$ is the set of natural numbers and $n.\infty = n$, for all $n \in \mathbb{N}$, *are called projection algebras*.
- An action preserving map between projection algebras is called a *projection map*.
- The category of projection algebras with projection maps between is denoted by **PRO**.

This notion in theoretical Computer scientists is used as a convenient means for algebraic specification of process algebras (see [6]).

Some of the algebraic and categorical properties of Projection algebras (or spaces) have been introduced and studied as an algebraic version of ultrametric spaces as well as algebraic structures, for example, in [8, 3].

By an ordered projection algebra we mean a projection algebra A which is also a poset such that the order is compatible with the action in the sense that the action preserves the order of the set and the usual order of natural numbers.

Pomonoid and S-poset

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Let S be a po-monoid. A *(right) S-poset* is a poset A which is also an S-set whose action $\lambda : S \times A \to A$ is order-preserving, where $A \times S$ is considered as a poset with componentwise order.

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An *S-poset map* (or *morphism*) is an action preserving monotone map between S-posets.

Ordered projection algebras

- We call S-posets, for the pomonoid $S = (\mathbb{N}^\infty, \min, \leq)$, *ordered projection algebras*.
- We denote the category of ordered projection algebras with monotone projection maps between them by **O-PRO**.

Cont. Introduction and Preliminaries

In general, S-posets, posets with an action of a partially order monoid S on them have been studied for example in [1].

In [4, 5], it is shown that *the only injective S-posets with respect to monomorphisms are trivial ones*, but *there are enough* injective S-posets with respect to regular monomorphisms, called *regular injective S-posets*.

Regular monomorphism and Order-embedding

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- Monomorphisms in the category of S-posets are exactly the injective morphisms.
- Regular monomorphisms (morphisms which are equalizers) are exactly *order-embeddings*; that is, S-poset maps $f : A \to B$ for which we have:

$$f(a) \leq f(a') \text{ if and only if } a \leq a'$$

for all $a, a' \in A$.

Regular injectivity and ideal injectivity

Now,

- we study injectivity of ordered projection algebras with respect to order-emdedding projection maps, and
- compare it with injectivity with respect to the embeddings of the form $I \to \mathbb{N}^{\infty}$ for a poideal I of $\mathbb{N}^{\infty}$.

Following S-posets, we call an ordered projection algebra A *regular injective* if $\forall$ order-embedding projection map $f : B \to C$ and $\forall$ monotone projection map $g : B \to A$, $\exists$ a monotone projection map $h : C \to A$ such that $hf = g$:

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[dr, "g"'] & C \arrow[d, "h"] \\ & A \end{tikzcd}$$

Ideal

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- Notice that ideals of $\mathbb{N}^\infty$ are of the form $\downarrow k = \{n \in \mathbb{N} : n \leq k\}$ for $k \in \mathbb{N}^\infty$ and the set $\mathbb{N}$.

Ideal injectivity

For an ideal I of $\mathbb{N}^\infty$, an ordered projection algebra A is called *I-injective*, if all monotone projection map $f : I \to A$ can be extended to $\mathbb{N}^\infty$.
An ordered projection algebra A is said to be *ideal injective*, if it is I-injective, for all ideal I of $\mathbb{N}^\infty$.

$$\begin{tikzcd} I \arrow[r, "\imath"] \arrow[dr, "f"'] & \mathbb{N}^\infty \arrow[d, "\overline{f}"] \\ & A \end{tikzcd}$$

$\mathbb{N}$-injectivity

Theorem

Every ordered projection algebras is $\downarrow k$-injective, for $k \in \mathbb{N}$.

Proof: Extend a morphism $f : {\downarrow}k \to A$ to $\mathbb{N}^{\infty}$ by defining $\overline{f}(n) = nf(k)$.

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Corollary

For ordered projection algebras, ideal injectivity coincides with $\mathbb{N}$-injectivity.

Theorem

For ordered projection algebras, the following are equivalent:

(1) *Ideal injectivity in* **O-PRO**.

(2) $\mathbb{N}$*-injectivity in* **O-PRO**.

(3) $\mathbb{N}$*-injectivity in* **PRO**.

(4) *injectivity in* **PRO**.

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(4) *injectivity in* **PRO**.

Proof: (1) and (2) are equivalent by the above corollary.
(2)⇔(3) By applying the above Lemma.
(3)⇔(4) It has been proved in [3]:

Corollary

Regular injectivity in **O-PRO** *implies injectivity in* **PRO**. *But the converse is not true.*

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Regular injectivity in **O-PRO** *implies injectivity in* **PRO***. But the converse is not true.*

Proof: It is clear that regular injectivity in **O-PRO** implies ideal injectivity, and so we have the result by applying the above theorem.
To see that the converse is not true, take an injective projection algebra and consider it with the natural order, then it is not regular injective as we see later on!

Blocks

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We call equivalence classes of $\sim$, the *blocks* of A.
Notice that the quotient ordered projection algebra $A/\sim$ of all blocks has the order $[a] \leq [b] \Leftrightarrow 1a \leq 1b$.

Properties of Blocks

- Each block of an ordered projection algebra is a subalgebra and has just one fixed element.
  Also, if nx, for some $n \in \mathbb{N}$ belongs to a block then x belongs to that block.

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- Each block of an ordered projection algebra is a subalgebra and has just one fixed element.
  Also, if nx, for some $n \in \mathbb{N}$ belongs to a block then x belongs to that block.
- If an ordered projection algebra A has a top element T which is also a fixed element, then the block containing T is $\{T\}$.
- For each block $[a]_\sim$ and $x \in [a]$, $\mathbb{N}x$ is the countable bounded chain:

$$1a = 1x \leq 2x \leq nx \leq \cdots \leq x.$$

Lemma

For any ordered projection algebra A, $A/\sim$ is isomorphic to $Fix(A)$.

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Proof.

The assignment $[a] \mapsto 1a$ is the required isomorphism. $\square$

Theorem

If A is a regular injective ordered projection algebra then so is $Fix(A)$. The converse is not generally true.

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Proof: There is a retraction $f : A \to Fix(A)$ given by $f(a) = 1a$. Then $Fix(A)$ being a retract of a regular injective ordered projection algebra, is itself regular injective.
For the converse, take a one-fixed non trivial ordered projection algebra A (such as $\mathbb{N}$ or $\mathbb{N}^\infty$). It is not regular injective since it does not have two fixed elements (see [4]), but $Fix(A)$ is clearly regular injective.

Regular injectivity and Blocks

Theorem

If A is a regular injective ordered projection algebra then each block $[a]$ is complete as a poset, and is injective as a projection algebra.

Proof

If A is regular injective then it is injective in **PRO**. This gives that each block is also injective in **PRO**, because each projection map $f : \mathbb{N} \to [a]$ can be extended $f : \mathbb{N}^\infty \to A = \coprod_{a \in A}[a]$. Then $\overline{f}$ factors through $[a]$. This is because, $n\overline{f}(\infty) = \overline{f}(n) = f(n) \in [a]$, for all $n \in \mathbb{N}$, implies $\overline{f}(\infty) \in [a]$.

$$\begin{tikzcd}
\mathbb{N} \arrow[r, "\imath"] \arrow[d, "f"'] & \mathbb{N}^\infty \arrow[dl, "\overline{f}"] \arrow[d, "\overline{f}"] \\
{[a]} \arrow[r, "\imath"'] & A = \coprod_{a \in A}[a]
\end{tikzcd}$$

Cont. Proof

Also, it is shown (see [5]) that A being regular injective, is continuously complete (that is, is complete and supremum maps preserve the action: $n \bigvee X = \bigvee nX$, for all $n \in \mathbb{N}^{\infty}$).

This gives that each block is also continuously complete. This is because, if $Y \subseteq [a]$ then $\bigvee Y$, which exists in A, belongs to $[a]$, since $1 \bigvee Y = \bigvee_{y \in Y} 1y = 1a$.

Naturally ordered projection algebras

- The *natural order* on a projection algebra A is the order given by $a \leq b$ if and only if $a = nb$, for $n \in \mathbb{N}^{\infty}$.
- Notice that the pomonoid $S = \mathbb{N}^{\infty}$ is itself naturally totally ordered, since $m \leq n \Leftrightarrow m = m.n$.

  We denote the category of naturally ordered projection algebras by $\mathbf{O\text{-}PRO_{nat}}$.

Lemma

If A is a naturally ordered projection algebra then for every ordered projection algebra B,

$$Hom_{\mathbf{PRO}}(A, B) = Hom_{\mathbf{O-PRO}}(A, B).$$

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Proof.

Let $f : A \to B$ be a projection map, and $a \leq a'$ in A. Then there exists $n \in \mathbb{N}$ with $a = na'$. Thus, $f(a) = f(na') = nf(a') \leq \infty f(a') = f(a')$, and hence $f(a) \leq f(a')$. □

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The category **PRO** *is isomorphic to the category* $\mathbf{O\text{-}PRO_{nat}}$.

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Proof.

Given a projection algebra, the natural order makes it into an ordered projection algebra. Also, all projection maps between projection algebras preserve the natural order. This assignment is functorial, and the obtained functor is clearly an isomorphism whose inverse functor is the forgetful functor. □

Naturally ordered projection algebras and injectivity

Lemma

A naturally ordered projection algebra A is injective as an object in **PRO** *if and only if A is injective in* $\mathbf{O\text{-}PRO_{nat}}$.

Theorem

A continuously complete naturally ordered projection algebra A is injective in $\mathbf{O\text{-}PRO_{nat}}$. But, the converse is not generally true.

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A continuously complete naturally ordered projection algebra A is injective in $\mathbf{O\text{-}PRO_{nat}}$. *But, the converse is not generally true.*

Proof.

Applying last lemma, we prove that a continuously complete projection algebra A is injective in **PRO**. To see this, we prove that A is $\mathbb{N}$-injective.
Let $f : \mathbb{N} \to A$ be a projection map. We must extend it to a projection map $\overline{f} : \mathbb{N}^{\infty} \to A$.
Define $a = \bigvee f(\mathbb{N})$, and then $\overline{f}(\infty) = a$, and of course $\overline{f}(n) = f(n)$ for $n \in \mathbb{N}$.
To prove that, $\overline{f}$ is a projection map, it is enough to prove that $\overline{f}(n) = n\overline{f}(\infty)$. We have,

$$n\overline{f}(\infty) = na = n\bigvee f(\mathbb{N}) = n\bigvee_{m\in\mathbb{N}} f(m) = \bigvee_{m\in\mathbb{N}} f(nm).$$

$\square$

Cont. Proof

But $\bigvee_{m\in\mathbb{N}} f(nm) = f(n)$, since $f(n) = f(nn) \leq \bigvee_{m\in\mathbb{N}} f(nm)$. Also, for each m, $f(nm) = mf(n) \leq \infty f(n) = f(n)$ which gives $\bigvee_{m\in\mathbb{N}} f(nm) \leq f(n)$. Thus A injective in **PRO** as required.

To see that the converse fails, take $A = \mathbf{1} \sqcup \mathbf{1}$. Then A is injective in **PRO**, and hence in $\mathbf{O\text{-}PRO_{nat}}$. But A being a non complete poset, is not continuously complete.

Cyclic ordered projection algebras

Definition

An ordered projection algebra A of the form $A = \mathbb{N}^{\infty} a$, for some $a \in A$, is called *cyclic*.

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An ordered projection algebra A of the form $A = \mathbb{N}^{\infty}a$, for some $a \in A$, is called *cyclic*.

Notice that a cyclic ordered projection algebra has necessarily natural order and is a countable bounded chain.
In fact, it is of the form $\{1a, 2a, 3a, \cdots, a\}$, with $1a \leq 2a \leq 3a \leq \cdots \leq a$.

Cyclic ordered projection algebras and injectivity

Theorem

For a projection algebra A with natural order, the following are equivalent:
(1) *A is a complete poset.*
(2) *A is a continuously complete ordered projection algebra.*
(3) *A is an infinite countable bounded chain.*
(4) *A is an infinite countable complete chain.*
(5) *A is a cyclic projection algebra.*

Proof

We prove (1) $\Rightarrow$ (5) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (2) $\Rightarrow$ (1).
(1) $\Rightarrow$ (5) Let the natural ordered projection algebra A be complete as a poset. Then A is cyclic, generated by $a = \bigvee A$. This is because, for $x \in A$ we have $x \leq \bigvee A = a$, and hence there exists $n \in \mathbb{N}$ with $x = na$, and then $x \in \mathbb{N}^\infty a$.
(5) $\Rightarrow$ (3) is true by description of cyclic (ordered) projection algebras.
(3) $\Leftrightarrow$ (4) The part $\Leftarrow$ is clear. For the converse, notice that $A \cong \mathbb{N}^\infty$ and $\mathbb{N}^\infty$ is complete. In fact, for finite $X \subseteq \mathbb{N}^\infty$, $\bigvee X = maxX$, and $\bigwedge X = minX$; and for infinite $X \subseteq \mathbb{N}^\infty$, $\bigvee X = \infty$. For the latter, notice that if $\infty \neq a$ is an upper bound of X, then $X \subseteq {\downarrow}a$ and so X is finite.

Cont. Proof

(4) $\Rightarrow$ (2) Take $A = \mathbb{N}^{\infty}$, $n \in \mathbb{N}$, and $X \subseteq A$. If $\infty \in X$ then the result is clear, so we assume that $X \subseteq \mathbb{N}$. We have to show that
$n(\bigvee X) = \bigvee_{x \in X} nx$.
If X is finite, then $\bigvee X = maxX \in \mathbb{N}$, also notice that $\{nx : x \in X\} \subseteq {\downarrow}n$.
We consider two cases: (1) $n \in X$; (2) $n \notin X$.
If $n \in X$, then $n \leq \bigvee X$, and so $n \bigvee X = n$. Also,
$n = nn \in \{nx : x \in X\} \subseteq {\downarrow}n$ gives $\bigvee_{x \in X} nx = n$.

Cont. Proof

If $n \notin X$, then we consider two cases:

$$\text{(a) } x \leq n, \forall x \in X, \qquad \text{(b) } \exists x_0 \in X, n < x_0$$

In case (a), $\bigvee X \leq n$, and so $n \bigvee X = \bigvee X$. Also,
$\bigvee_{x \in X} nx = \bigvee_{x \in X} x = \bigvee X$.
In case (b), $nx_0 = n$ and so $\bigvee_{x \in X} nx = n$. Also, $x_0 \leq \bigvee X$ implies $n = nx_0 \leq n \bigvee X \leq n$, which means $n \bigvee X = n$.
If X is infinite, then $\bigvee X = \infty$ (see the proof of (3) $\Leftrightarrow$ (4)). Therefore, $n \bigvee X = n$. Also, since X is finite, there exists $x_0 \in X$, $nx_0 = n$, since otherwise $x \leq n$ for all $x \in X$ which means X is finite. Now, with a similar discussion as part (4) $\Rightarrow$ (2), we have $\bigvee_{x \in X} nx = n$.
(2) $\Rightarrow$ (1) is clear.

Remark

Notice that the assumption that A being naturally ordered is not redundant from the above theorem.

For example, the three element chain $x_0 \leq x_1 \leq T$ such that x_0 and T are fixed elements, and $nx_1 = x_0$ for all $n \in \mathbb{N}$ is continuously complete, but it is not cyclic, since $\mathbb{N}^{\infty}x_1 = \{x_0, x_1\}$.

Corollary

A naturally ordered projection algebra satisfying one of the equivalent conditions of the above theorem is injective in **PRO**.

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Example

The above corollary shows that $\mathbb{N}^{\infty}$ is injective in **PRO**. Also, as we saw this gives that it is also ideal injective in **O-PRO**. Notice that, it is not regular injective, because it does not have two fixed elements.

Regular injectivity of naturally ordered projection algebras

Proposition

There exist no non trivial regular injective projection algebra with natural order in **O-PRO**.

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There exist no non trivial regular injective projection algebra with natural order in **O-PRO**.

Proof: Let A be a non trivial regular injective projection algebra with natural order. Then A is bounded with two zero elements, namely $0 \leq 1$. Since the order is natural, this gives some $n \in \mathbb{N}$ such that $n0 = 1$. This contradicts the fact that 0 is a fixed element, and it is distinct from 1.

Bare Criterion for injectivity

Theorem (Baer Criterion)

Let A be an ordered projection algebra.
(1) *If A is injective as an object of* **PRO** *then A is injective with respect to ordered projection algebras with natural order.*
(2) *If A is injective as an object of* **PRO** *with respect to one fixed projection algebras* $($*with natural order*$)$ *then A is injective with respect to all projection algebras (with natural order).*
(3) *If A is regular injective with respect to embeddings into cyclic* **O-PRO** *then A is regular injective with respect to all projection algebras with natural order.*

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Thank you