Duality for Relations on Ordered Algebras

M. Andrew Moshier[1] Alexander Kurz[2]

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Chapman University

University of Leicester

The Basic Motivation

In algebraic logic

- Peter Jipsen and Nick Galatos' talks yesterday refered to the idea of weakening relations on a poset.
- The same idea works in Priestley spaces – the duals of distributive lattices.
- So it should be possible to study *relations* on ditributive lattices via Priestley weakening relations.
- Generally, we seek to understand the general setting in which relation lifting carries over in natural dualities.

Ordered Algebras

A class of algebras for a given signature is ordered if

- The category is concrete over Pos – there is a forgetful functor to Pos (and there are free algebras over posets); and
- All operations in the signature are monotonic.

Examples

- Distributive lattices
- Meet semilattices
- Frames (signature is infinitary)
- Complemented distributive algebras.

Non-examples

- Heyting algebras
- Boolean algebras

Priestley Structures

Analogous definitions work for expansions of Priestley spaces.

- Operations are continuous and monotonic
- Relations are topologically closed, and weakening closed (a subtlety here for relations of arity >2 that won't concern us today).

Example

- Priestley distributive lattices. These are the duals of posets (Banaschewski).

Relations Three Ways

In the following "poset" means either poset *simpliciter* or Priestley space (poset with discrete topology versus with a Priestley sepatated Stone topology)

Spans

For posets X and Y, a span from X to Y is a pair of monontic functions

$$X \xleftarrow{p} P \xrightarrow{q} Y$$

$\mathrm{Span}(X, Y)$ is the category of spans from X to Y.

A morphism from $X \xleftarrow{p} R \xrightarrow{q} Y$ to $X \xleftarrow{p'} R' \xrightarrow{q'} Y$ is a monotonic function $f{:}R \to R'$ making the obvious triangles commute.

Relations Three ways

Cospans

For posets X and Y, a cospan rom X to Y is a pair of morphisms

$$X \xrightarrow{j} C \xleftarrow{k} Y$$

Cospan(X, Y) is the category of cospans from X to Y.

A morphism from $X \xrightarrow{j} C \xleftarrow{k} Y$ to $X \xrightarrow{j'} C' \xleftarrow{k'} Y$ is a monotonic function $f{:}C \to C'$ making the obvious triangles commute.

Relations

Weakening relations

For posets X and Y, a weakening relation is monotonic map $R\colon X^{\partial} \times Y \to 2$. Equivalently, identifying with the cokernel $R = \{(x, y) \mid R(x, y) = 1\}$:

$$\frac{x \leq_X x' \qquad x' \; R \; y' \qquad y' \leq_X y}{x \; R \; y}$$

WRel(X, Y) is the poset (regarded as a category) of weakening relations order pointwise.

How these are related?

Weakening relations, spans and cospans are related via adjunctions.

- $R \in \mathrm{WRel}$, determines
  - a span graph(R) by restricting projections
  - a cospan collage(R) by taking the least order on $X \uplus Y$ containing $\leq_X$, $\leq_Y$ and R
- $X \xleftarrow{p} R \xrightarrow{q} Y$ determines
  - a weakening relation $\mathrm{rel}_s(p,q)$ by (x,y) iff $\exists r \in R$, $x \leq p(r)$ and $q(r) \leq y$
  - a cospan cocomma(p,q) by taking the cocomma of (p,q) – the order analogue of a push out.
- $X \xrightarrow{j} C \xleftarrow{k} Y$ determines
  - a weakening relation $\mathrm{rel}_c(j,k)$ by (x,y) iff $j(x) \leq k(y)$
  - a span comma(j,k) by taking the comma of (j,k) – the order analogue of a pull back.

How are these related?

All three are 2-categories.
We already described the hom categories: $mathsf Span(X,Y)$, $mathsf Cospan(X,Y)$ and $\mathrm{WRel}(X,Y)$.

- Composition of spans is defined by a comma
- Composition of cospans is defined by a cocomma
- Composition of weakening relations is defined by relational product: $R;S(x,y) = \bigvee_{y \in Y} R(x,y) \wedge S(y,z)$.

How are these related?

So rel_s, rel_c, graph, etc., are two functors and

- $\mathrm{rel}_s \dashv \mathrm{graph}$ and $\mathrm{graph} \circ \mathrm{rel}_s = \mathrm{Id}$
- $\mathrm{rel}_c \dashv \mathrm{collage}$ iand $\mathrm{collage} \circ \mathit{relcf} = \mathrm{Id}$.
- cocomma $\dashv$ comma
- $\mathrm{comma} \cong \mathrm{graph} \circ \mathrm{rel}_c$.
- $\mathrm{cocomma} \cong \mathrm{collage} \circ \mathrm{rel}_s$.

All these facts hold analogously in PoSpace, the category of topological spaces with closed partial orders. Definitions are with respect to continuous montonic functions.

The bottom line

We also characterize those spans and cospans that arise as graphs and collages of weakening relations.

These are the same as those that are fixed by comma $\circ$ cocomma or cocomma $\circ$ comma.

Extending to algebras and topological structures

Suppose $\mathcal{A}$ is a class of ordered algebras.
Let $\overline{\mathcal{A}}$ denote the category of $\mathcal{A}$-algebras spans in $\mathcal{A}$ with weakening poset reducts.

For example, $\overline{\text{DLat}}$ is the category of bounded distributive lattices with morphisms that are relations satisfying:

- $x \leq x' \ R \ y' \leq y$ implies $x \ R \ y$
- $0 \ R \ y$ y
- $x \ R \ 1$
- $x \ R \ y_0$ and $x \ y_1$ implies $x \ R \ y_0 \wedge y_1$
- $x_0 \ R \ y$ and $x_1 \ R \ y$ implies $x_0 \vee x_1 \ R \ y$.

Bringing it home

Theorem

- *$\overline{DL}$ is (co)dually equivalent to $\overline{\text{Priestley}}$.*
- *$\overline{Pos}$ is (co)dually equivalent to $\overline{\text{Stone}(DLat)}$*

Proof idea: A span $X \xleftarrow{p} R \xrightarrow{q} Y$ in DLat dualizes to $2^X \xrightarrow{2^p} 2^R \xleftarrow{2^q} 2^Y$ in Priestley.

But this transfer preserves the weakening property.

The correspondence of spans and cospans allows this cospan in Priestley to be turned into a span.

The second claim comes from swapping the Stone topology and discrete topology in the first claim.