Undecidability of $\{\cdot, 1, \lor\}$-equations in subvarieties of commutative residuated lattices.

Gavin St. John

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A (commutative) **residuated lattice** is a structure \( \mathbb{R} = (R, \cdot, \lor, \land, \setminus, /, 1) \), such that

- \((R, \lor, \land)\) is a lattice
- \((R, \cdot, 1)\) is a (commutative) monoid
- For all \(x, y, z \in R\)

\[
x \cdot y \leq z \iff y \leq x \setminus z \iff x \leq z / y,
\]

where \(\leq\) is the lattice order.

We denote the variety of (commutative) residuated lattices by \((CRL)\) RL.

If \((r)\) is a rule (axiom), then \((C)RL_r := (C)RL + (r)\).
Known results for Quasi-Equational Theory

\((k^m_n)\) represents the knotted rule \(x^n \leq x^m\)

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Consequently, extensions of $\mathcal{CRL}$ in the signatures $\{\leq, \cdot, 1\}$ have been fully characterized.

We inspect (in)equations in the signature $\{\cdot, 1, \lor\}$.

Proof theoretically, such axioms correspond to inference rules, e.g.,

$$x \leq x_2 \lor 1 \iff X,Y,Y,Z \vdash C \quad X,Z \vdash C \quad X,Y,Z \vdash C$$

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Linearization

Any equation \( s = t \) in the signature \( \{\cdot, 1, \lor\} \) is equivalent to some conjunction of linear inequations we call “d-rules” of the form:

\[
(d) \quad x_1 \cdots x_n \leq \bigvee_{j=1}^{m} x_1^{d_j(1)} \cdots x_n^{d_j(n)},
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where \( d := \{d_1, \ldots, d_m\} \subset \mathbb{N}^n \).
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where \( d := \{d_1, \ldots, d_m\} \subset \mathbb{N}^n \). Such conjoins can be determined by the properties of \( \mathcal{CRL} \):

- \( x \leq y \iff x \lor y = y \)
- \( x \lor y \leq z \iff x \leq z \text{ and } y \leq z \)
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- $x \leq y \iff x \lor y = y$
- $x \lor y \leq z \iff x \leq z \text{ and } y \leq z$
- **linearization**
  
  E.g., the rule

  $$(\forall u)(\forall v) \quad u^2v \leq u^3 \lor uv$$

  is equivalent to, via the substitutions $u = x \lor y$ and $v = z$,

  $$(\forall x)(\forall y)(\forall z) \quad xyz \leq x^3 \lor x^2y \lor xy^2 \lor y^3 \lor xz \lor yz$$
Conditions on $d \subset \mathbb{N}^n$

- If $(d)$ implies a knotted rule, then $\mathcal{CRL} + (d)$ is decidable.
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- We view $d = \{d_j\}_{j=1}^m$ as a set of linear subspaces of $\mathbb{R}^n$. 

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\((\star)\) Given any nonempty \(A \subseteq \{1, ..., n\}\), and any nontrivial valuation of variables \(x_1, ..., x_n\) in \(\mathbb{N}\), there exists \(j \neq j' \leq m\) such that the supports of \(d_j\) and \(d_{j'}\) intersect \(A\), and

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  $$\sum_{i=1}^n d_j(i)x_i \neq \sum_{i=1}^n d_{j'}(i)x_i$$

  $(\star\star)$ For any valuation of the $x_i$'s, there exists $j \leq m$ such that
  $$\sum_{i=1}^n x_i < \sum_{i=1}^n d_j(i)x_i$$
## Examples and Non-examples of \((\ast)\) & \((\ast\ast)\)

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Determining whether a given (d)-rule satisfies these conditions amounts to showing certain systems of equations do not have “non-trivial ” solutions in $\mathbb{N}^n$. This can be simplified by asking if there are positive solutions in $\mathbb{R}^n$. 

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- a finite set $Q$ of states with designated initial state $q_I$ and final state $q_f$,
- and a finite set $P$ of instructions $p$ of the form:
  - Increment: $q \leq^p q' r$
  - Decrement: $q r \leq^p q'$
  - Fork: $q \leq^p q' \lor q''$

where $q, q', q'' \in Q$ and $r \in R_k$. 
Instructions of an ACM act on configurations, which consist of a single state and a number register tokens

\[ C = qr_1^{n_1}r_2^{n_2} \cdots r_k^{n_k}. \]
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Forking instructions allow parallel computation. The status of a machine at a given time in a computation is called an **instantaneous description** (ID),

\[ u = C_1 \lor C_2 \lor \cdots \lor C_n, \]

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An instruction \( p \) acts on a single configuration of an ID \( u \) to create a new configuration \( u' \).
We view computations as order relations on the free commutative idempotent semiring $A_M = (A_M, \lor, \cdot, \bot, 1)$ generated by $Q \cup R_k$, where $M = (R_k, Q, P)$ is a $k$-ACM and
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Each instruction $p \in P$ defines a relation $\leq^p$ closed under

$$\frac{u \leq^p v}{ux \leq^p vx} \quad [\cdot] \quad \text{and} \quad \frac{u \leq^p v}{u \lor w \leq^p v \lor w} \quad [\lor],$$

for $u, v, w \in \text{ID}(M)$ and $x \in R_k^*$, where $R_k^*$ is the free commutative monoid generated by $R_k$. 

Computations

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for $u, v, w \in \text{ID}(M)$ and $x \in R_k^*$, where $R_k^*$ is the free commutative monoid generated by $R_k$.

We define the computation relation $\leq_M$ to be the smallest preorder containing $\bigcup_{p \in P} \leq^p$. 
We say a machine $M$ terminates on an ID $u$ if $u \leq_M q_f$. 

Example Machine

Let $M = M_{\text{even}} := (\{r\}, \{q_0, q_1, q_f\}, \{p_1, p_2, p_3\})$, with instructions

$q_0 r \leq p_1 q_1$;
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- If $u = C_1 \lor \cdots \lor C_n$, then $u \leq_M q_f$ iff $C_i \leq_M q_f$, $\forall i \leq n$. 

Example Machine

Let $M = M_{\text{even}} := (\{r\}, \{q_0, q_1, q_f\}, \{p_1, p_2, p_3\})$, with instructions

$q_0 \rightarrow r \leq p_1 q_1$;
$q_1 \rightarrow r \leq p_2 q_0$;
$q_0 \leq p_3 q_f$.

Note that $q_0 \rightarrow r_n \leq M q_f$ iff $n$ is even.
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- If $u = C_1 \lor \cdots \lor C_n$, then $u \leq_M q_f$ iff $C_i \leq_M q_f$, $\forall i \leq n$.
- If $u \leq_M q_f$, then there exists $p_1, \ldots, p_n \in P$ and $u_0, \ldots, u_n \in \text{ID}(M)$, such that 
  $$u = u_0 \leq^{p_1} u_1 \leq^{p_2} \cdots \leq^{p_n} u_n = q_f.$$
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$$q_0 r \leq^{p_1} q_1; \quad q_1 r \leq^{p_2} q_0; \quad q_0 \leq^{p_3} q_f.$$ 

- Note that $q_0 r^n \leq_M q_f$ iff $n$ is even.

$$q_0 r^4 \leq^{p_1} q_1 r^3 \leq^{p_2} q_0 r^2 \leq^{p_1} q_1 r \leq^{p_2} q_0 \leq^{p_3} q_f$$
We say a machine $M$ **terminates on an ID** $u$ if $u \leq_M q_f$.

- If $u = C_1 \lor \cdots \lor C_n$, then $u \leq_M q_f$ iff $C_i \leq_M q_f$, $\forall i \leq n$.
- If $u \leq_M q_f$, then there exists $p_1, \ldots, p_n \in P$ and $u_0, \ldots, u_n \in \text{ID}(M)$, such that

$$u = u_0 \leq_{p_1} u_1 \leq_{p_2} \cdots \leq_{p_n} u_n = q_f.$$ 

### Example Machine

Let $M = M_{\text{even}} := (\{r\}, \{q_0, q_1, q_f\}, \{p_1, p_2, p_3\})$, with instructions

$$q_0 r \leq_{p_1} q_1; \quad q_1 r \leq_{p_2} q_0; \quad q_0 \leq_{p_3} q_f.$$ 

- Note that $q_0 r^n \leq_M q_f$ iff $n$ is even.

\[
q_0 r^4 \leq_{p_1} q_1 r^3 \leq_{p_2} q_0 r^2 \leq_{p_1} q_1 r \leq_{p_2} q_0 \leq_{p_3} q_f \\
q_0 r^3 \leq_{p_1} q_1 r^2 \leq_{p_2} q_0 r \leq_{p_3} q_f r
\]
Theorem [Lincoln et. al., 1992]

There exists a 2-ACM $\tilde{M}$ such that membership of the set $\{u \in \text{ID}(\tilde{M}) : u \leq_{\tilde{M}} q_f\}$ is undecidable. Furthermore, it is undecidable whether $q_I \leq_{\tilde{M}} q_f$. 
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- Given an ACM $M$ we define the theory of $M$ $\text{Th}(M)$ to be
  the conjunction of all syntactic instructions in $P$, i.e.,
  \[
  \text{Th}(M) := \bigwedge \{ C \leq u : (C \leq^p u) \in P \}.
  \]
Theorem [Lincoln et. al., 1992]

There exists a 2-ACM $\widetilde{M}$ such that membership of the set
$\{ u \in ID(\widetilde{M}) : u \leq_{\widetilde{M}} q_f \}$ is undecidable. Furthermore, it is
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- Given an ACM $M$ we define the theory of $M$ $Th(M)$ to be the conjunction of all syntactic instructions in $P$, i.e.,
  $$Th(M) := \& \{ C \leq u : (C \leq^p u) \in P \}.$$  

- Given an ID $u$, we define the quasi-equation $Halt_M(u)$ to be
  $$Th(M) \implies u \leq q_f.$$
Given a d-rule, e.g. $[d]$ is given by $x \leq x^2 \lor x^4$, we add “ambient” instructions of the form

$$qxy \leq^d qxy^2 \lor qxy^4,$$

for each $q \in Q$ and any $x, y \in R^*_k$. 

\[d\text{-rules and the relation } \leq_d(M)\]
Given a d-rule, e.g. $[d]$ is given by $x \leq x^2 \lor x^4$, we add “ambient” instructions of the form

$$qxy \leq_d qxy^2 \lor qxy^4,$$

for each $q \in Q$ and any $x, y \in R^*_k$.

As with the instructions in $P$, we close $\leq_d$ under the inference rules $[\cdot]$ and $[\lor]$, and we define the relation $\leq_d(M)$ to be the smallest preorder generated by $\leq_d \cup \leq_M$. 

$d$-rules and the relation $\leq_d(M)$
Clearly, if $u \leq_M q_f$ then $u \leq_{d(M)} q_f$ since $\leq_M \subseteq \leq_{d(M)}$. 
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However, for some ACM’s $M$, it’s possible that $u \leq d(M) q_f$ but $u \nleq_M q_f$. 
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**Example**

Consider $M = M_{\text{even}}$ and (d) given by $x \leq x^2 \lor x^4$. 
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### Example

Consider \( M = M_{\text{even}} \) and \((d)\) given by \( x \leq x^2 \lor x^4 \).

- \( q_0 r^3 \not\leq_M q_f \) since 3 is odd.
Clearly, if $u \leq_M q_f$ then $u \leq_{d(M)} q_f$ since $\leq_M \subseteq \leq_{d(M)}$.

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**Example**

Consider $M = M_{\text{even}}$ and (d) given by $x \leq x^2 \lor x^4$.

- $q_0 r^3 \not\leq_M q_f$ since 3 is odd.
- However, $q_0 r^3 \leq_{d(M)} q_f$, witnessed by

$$q_0 r^3 = q_0 r^2 r \leq_d q_0 r^2 r^2 \lor q_0 r^2 r^4 = q_0 r^4 \lor q_0 r^6 \leq_{d(M)} q_f,$$

since $q_0 r^4 \leq_M q_f$ and $q_0 r^6 \leq_M q_f$. 
Given an ACM $M$ and a $d$-rule, is it possible to construct a new ACM $M'$ such that

$$u \leq_M q_f \text{ if and only if } \theta(u) \leq_{d(M')} q_F,$$

(where $\theta: \text{ID}(M) \rightarrow \text{ID}(M')$ is computable and $q_F$ is the final state of $M'$) and if so, under what conditions?
Then $M_K$ machine

Let $M = (R_2, Q, P)$ be a 2-ACM and let $K > 1$ be given. We define the 3-ACM $M_K = (R_3, Q_K, P_K)$ such that
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Let $M = (R_2, Q, P)$ be a 2-ACM and let $K > 1$ be given. We define the 3-ACM $M_K = (R_3, Q_K, P_K)$ such that

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- each increment and decrement instruction of $P$ is replaced by multiply and divide by $K$ programs, i.e.

$$
\begin{align*}
q & \leq^p q' r \in P \implies q r \sqcup^p q' r K \sqcup & \subset P_K \\
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  $$q \leq^p q' r \in P \implies qr \sqsupset^p q' r_K \sqsupset^p q' r_K' \in P_K,$$
  $$qr \leq^p q' \in P \implies qr \sqsupset^p q' r_K \sqsupset^p q' r_K' \in P_K.$$
- We obtain, for each $q \in Q$,
  
  $$qr_{1}^{n_1} r_{2}^{n_2} \leq_M q_f \iff qr_{1}^{K^{n_1}} r_{2}^{K^{n_2}} \leq_{M_K} q_F.$$
Detecting applications of $\leq^d$

**Observation**

Consider a configuration where the contents of some register $r$ is $n = s + t$, whereafter $\leq^d$ is applied to $t$-many tokens, i.e.,

$$qr^n = qr^s r^t \leq^d qr^s (r^{2t} \lor r^{4t}) = qr^{s+2t} \lor qr^{s+4t}$$
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q_r n = q_r s r^t \leq^d q_r s (r^{2t} \lor r^{4t}) = q_r s^{+2t} \lor q_r s^{+4t}
\]

Fact

For \( d : x \leq x^2 \lor x^4 \), if \( K \geq (4 - 2) + 1 = 3 \), it is impossible for \( s + 2t \) and \( s + 4t \) to both be powers of \( K \).
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**Fact**

For $d : x \leq x^2 \lor x^4$, if $K \geq (4 - 2) + 1 = 3$, it is **impossible** for $s + 2t$ and $s + 4t$ to both be powers of $K$.

- Such a $K$ will exist for any rule satisfying $(\star)$.
- Consequently, $qr^n \leq_d (M_K) qf$ iff $qr^n \leq_{M_K} qF$.
- For rules in more than one variable, satisfying $(\star\star)$ is sufficient to guarantee “detection.”
Let \( M = \overline{M} = (R_2, Q, P) \) be the 2-ACM such that it is undecidable whether \( q_I \leq_M q_f \). Consider the rule (d) be given by \( x \leq x^2 \lor x^4 \).
We construct \( M_K = (R_3, Q_K, P_K) \) for \( K = 3 \).
Let $M = \widetilde{M} = (R_2, Q, P)$ be the 2-ACM such that it is undecidable whether $q_I \leq_M q_f$. Consider the rule (d) be given by $x \leq x^2 \lor x^4$. We construct $M_K = (R_3, Q_K, P_K)$ for $K = 3$. By the observation, for any $q' \in Q_3$,

$$q' r_1^{n_1} r_2^{n_2} r_3^{n_3} \leq_{M_3} q_F \iff q' r_1^{n_1} r_2^{n_2} r_3^{n_3} \leq_{d(M_3)} q_F.$$
Let $M = \tilde{M} = (R_2, Q, P)$ be the 2-ACM such that it is undecidable whether $q_I \leq_M q_f$. Consider the rule (d) be given by $x \leq x^2 \lor x^4$. We construct $M_K = (R_3, Q_K, P_K)$ for $K = 3$. By the observation, for any $q' \in Q_3$,

$$q'r_1^{n_1}r_2^{n_2}r_3^{n_3} \leq_M q_f \iff q'r_1^{n_1}r_2^{n_2}r_3^{n_3} \leq d(M_3) q_f.$$ 

Hence, for any $q \in Q$,

$$qr_1^{n_1}r_2^{n_2} \leq_M q_f \iff qr_1^{3n_1}r_2^{3n_2} \leq d(M_3) q_f,$$

so it is undecidable whether $q_I r_1 r_2 \leq d(M_3) q_f$. 

Gavin St. John

Undecidability of $\{\cdot, 1, \lor\}$-equations in subvarieties of commutative residuated lattices.
Let $\mathcal{V} \subseteq CRL$ be a variety. We can show $\mathcal{V}$ has an undecidable word problem (and hence quasi-equational theory) if we can demonstrate

$$\mathcal{V} \models \text{Halt}_{d(M_K)}(qIr_1r_2) \iff qIr_1r_2 \leq_M q_f.$$ 

- If $\mathcal{V} \subseteq CRL$ then ($\iff$) is immediate.
- We use the theory of **Residuated Frames** (Galatos & Jipsen 2013) for a completeness of encoding to provide a model and valuation proving the contrapositive of ($\Rightarrow$), for varieties $\mathcal{V}$ satisfying certain conditions.
A **residuated frame** is a structure $\mathbf{W} = (W, W', N, \circ, \|, \|, 1)$, s.t.

- $(W, \circ, 1)$ is a monoid and $W'$ is a set.
- $N \subseteq W \times W'$, called the *Galois relation*, and
- $\| : W \times W' \to W'$ and $\| : W' \times W \to W'$ such that
- $N$ is a **nuclear**, i.e. for all $u, v \in W$ and $w \in W'$,
  $$(u \circ v) \ N \ w \iff u \ N \ (w \ \| \ v) \iff v \ N \ (u \ \| \ w).$$
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- $N$ is a nuclear, i.e. for all $u, v \in W$ and $w \in W'$,
  $$(u \circ v) \ N w \text{ iff } u \ N (w \| v) \text{ iff } v \ N (u \| w).$$

Define $\triangledown : \mathcal{P}(W) \to \mathcal{P}(W')$ and $\triangleleft : \mathcal{P}(W') \to \mathcal{P}(W)$ via
$$X^\triangledown = \{ y \in W' : \forall x \in X, xNy \}$$ and
$$Y^\triangleleft = \{ x \in W : \forall y \in Y, xNy \},$$ for each $X \subseteq W$ and $Y \subseteq W'$.

Then $(\triangledown, \triangleleft)$ is a Galois connection.

So $X \xrightarrow{\gamma_N} X^{\triangledown \triangleleft}$ is a closure operator on $\mathcal{P}(W)$.
Theorem [Galatos & Jipsen 2013]

\[ W^+ := (\gamma_N[\mathcal{P}(W)], \cup_{\gamma_N}, \cap, \circ_{\gamma_N}, \setminus, \parallel, \gamma_N(\{1\})) , \]

\[ X \cup_{\gamma_N} Y = \gamma_N(X \cup Y) \text{ and } X \circ_{\gamma_N} Y = \gamma_N(X \circ Y), \]

is a residuated lattice.

Proposition [Galatos & Jipsen 2013]

All simple rules are preserved by \((-)^+\).
Termination as a nuclear relation

Let $M = (R_k, Q, P)$ be a $k$-ACM and $W := (Q \cup R_k)^*$ be the free commutative monoid generated by $Q \cup R_k$. 

Gavin St. John

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**The frame \( W_M \)**

Similar to Chvalovský & Horčík (2016), we let \( W' := W \) and define the relation \( N_M \subseteq W \times W' \) via

\[
x \ N_M \ z \ \iff \ xz \leq_M q_f,
\]

for all \( x, z \in W \). Observe that, for any \( x, y, z \in W \),

\[
xy \ N_M \ z \ \iff \ xyz \leq_M q_f \iff x \ N_M yz.
\]

Since \( W \) is commutative it follows that \( N_M \) is nuclear.
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Since $W$ is commutative it follows that $N_M$ is nuclear.

Lemma

$W_M := (W, W', N_M)$ is a residuated frame, $W^+ \in CR\mathcal{L}$, and there exists a valuation $\nu : Fm \to W^+$ such that $W^+, \nu \models \text{Th}(M)$. 
Lemma

Let \((d)\) be any rule satisfying \(\ast\). Define \(W_{d(M)} := (W, W', N_{d(M)})\). Then \(W_{d(M)}^+ \in CR\mathcal{L}_d\).
Lemma

Let \((d)\) be any rule satisfying \((\star)\). Define \(W_{d(M)} := (W, W', N_{d(M)})\). Then \(W_{d(M)}^+ \in CRL_d\).

Fix \(M = \tilde{M}\) be the 2-ACM such that it is undecidable whether \(q_I \leq_M q_f\).

Theorem

Let \((d)\) be a rule satisfying \((\star)\) and \((\star\star)\), and let \(K \geq 2\) be sufficiently large. Then it is undecidable whether \(W_{d(M_K)}^+ \models \text{Halt}_{\tilde{M}_K}(q_I r_1 r_2)\).
**Lemma**

Let (d) be any rule satisfying (\(\star\)). Define \(W_{d(M)} := (W, W', N_{d(M)})\). Then \(W_{d(M)}^+ \in \mathcal{CRL}_d\).

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**Theorem**

Let (d) be a rule satisfying (\(\star\)) and (\(\star\)\(\star\)), and let \(K \geq 2\) be sufficiently large. Then it is undecidable whether \(W_{d(M_K)}^+ \models \text{Halt}_{M_K}(q_I r_1 r_2)\).

**Corollary**

For any variety \(\mathcal{V} \subseteq \mathcal{CRL}\), if

\[
W_{d(M_K)}^+ \in \mathcal{V},
\]

then \(\mathcal{V}\) has an undecidable word problem, and hence an undecidable quasi-equational theory.
(k^m_n) represents the knotted rule x^n \leq x^m

<table>
<thead>
<tr>
<th>Undecidable Eq. Theory</th>
<th>Decidable Eq. Theory</th>
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<tbody>
<tr>
<td>RL + (k^m_n), 1 \leq n &lt; m</td>
<td>RL</td>
</tr>
<tr>
<td>CRL + (?)</td>
<td>CRL + (k^m_n)</td>
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</table>
We can encode the instructions of an ACM $M = (R_k, Q, P)$ as a single term $\theta_M$ using the full signature of of $\mathcal{CRL}$ via

$$\theta_M := 1 \land \bigwedge_{(C \leq_M u) \in P} C \rightarrow u.$$ 

Let (d) be given such that there exists $n \geq 1$ and $k, c_1, \ldots, c_n \geq 1$ such that

$$\mathcal{CRL}_d \models x^k \leq \bigvee_{i=1}^{n} x^{k+c_i}, \quad (\star \star \star)$$

then (d) can be used to “bootstrap” the undecidability of the quasi-equation theory of $\mathcal{CRL}_d$ to the equational theory.
Corollary

Let \((d)\) be a rule satisfying \((\star)\), \((\star\star)\), \((\star\star\star)\) and let \(K \geq 2\) be sufficiently large. Then it is undecidable whether

\[
\text{CRL}_d \models \theta_{MK} \rightarrow (qIr_1r_2 \rightarrow q_F),
\]

and therefore \(\text{CRL}_d\) has an undecidable equational theory.
Thank You!


