Undecidability of $\{\cdot, 1, \vee\}$-equations in subvarieties of commutative residuated lattices.

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Residuated Lattices

Definition

A (commutative) **residuated lattice** is a structure
$\mathbf{R} = (R, \cdot, \vee, \wedge, \backslash, /, 1)$, such that

- $(R, \vee, \wedge)$ is a lattice
- $(R, \cdot, 1)$ is a (commutative) monoid
- For all $x, y, z \in R$

$$x \cdot y \leq z \iff y \leq x \backslash z \iff x \leq z / y,$$

  where $\leq$ is the lattice order.

We denote the variety of (commutative) residuated lattices by ($\mathcal{CRL}$) $\mathcal{RL}$.
If (r) is a rule (axiom), then $(\mathcal{C})\mathcal{RL}_{\mathrm{r}} := (\mathcal{C})\mathcal{RL} + (\mathrm{r})$.

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  ○ Proof theoretically, such axioms correspond to inference rules, e.g.,

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  ∘ So we restrict our investigation to the commutative case.

Linearization

Any equation $s = t$ in the signature $\{\cdot, 1, \vee\}$ is equivalent to some conjunction of linear inequations we call "d-**rules**" of the form:

$$(\mathrm{d}) \quad x_1 \cdots x_n \leq \bigvee_{j=1}^{m} x_1^{d_j(1)} \cdots x_n^{d_j(n)},$$

where $\mathrm{d} := \{d_1, ..., d_m\} \subset \mathbb{N}^n$.

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- $x \leq y \iff x \vee y = y$
- $x \vee y \leq z \iff x \leq z$ and $y \leq z$
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  E.g., the rule
  $$(\forall u)(\forall v)\ u^2 v \leq u^3 \vee uv$$
  is equivalent to, via the substitutions $u = x \vee y$ and $v = z$,
  $$(\forall x)(\forall y)(\forall z)\ xyz \leq x^3 \vee x^2 y \vee xy^2 \vee y^3 \vee xz \vee yz$$

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- We view $\mathrm{d} = \{d_j\}_{j=1}^m$ as a set of linear subspaces of $\mathbf{R}^n$. $(\star)$ Given any nonempty $A \subseteq \{1, ..., n\}$, and any nontrivial valuation of variables $x_1, ..., x_n$ in $\mathbb{N}$, there exists $j \neq j' \leq m$ such that the supports of d_j and $d_{j'}$ intersect A, and
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$(\star\star)$ For any valuation of the x_i's, there exists $j \leq m$ such that
$$\sum_{i=1}^{n} x_i < \sum_{i=1}^{n} d_j(i)x_i$$

Examples and Non-examples of $(\star)$ & $(\star\star)$

Rule	$(\star)$	$(\star\star)$
$x \leq x^2$		✓
$x \leq x^2 \vee 1$		✓
$x \leq x^2 \vee x^3$	✓	✓
$xy \leq x^2 \vee y^2$		
$xy \leq x \vee x^2 y$		
$xy \leq x \vee x^2 y \vee y^2$	✓	✓
$xyz \leq x^3 \vee x^2 y \vee y^3 \vee y^2 z \vee z^3 \vee z^2 x$	✓	
$xyzw \leq x^2 yzw \vee x^3 y^2 z^2 w^2$	✓	✓

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$xy \leq x \vee x^2 y \vee y^2$	✓	✓
$xyz \leq x^3 \vee x^2 y \vee y^3 \vee y^2 z \vee z^3 \vee z^2 x$	✓	
$xyzw \leq x^2 yzw \vee x^3 y^2 z^2 w^2$	✓	✓

Determining whether a given (d)-rule satisfies these conditions amounts to showing certain systems of equations do not have "non-trivial " solutions in $\mathbb{N}^n$. This can be simplified by asking if there are positive solutions in $\mathbb{R}^n$.

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- a finite set Q of **states** with designated **initial state** q_I and **final state** q_f,
- and a finite set P of **instructions** p of the form:

  - **Increment:** $q \leq^p q'r$
  - **Decrement:** $qr \leq^p q'$
  - **Fork:** $q \leq^p q' \vee q''$,

  where $q, q', q'' \in Q$ and $r \in R_k$.

ACM's continued

- Instructions of an ACM act on **configurations**, which consist of a single state and a number register tokens
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- Forking instructions allow parallel computation. The status of a machine at a given time in a computation is called an **instantaneous description** (ID),

$$u = C_1 \vee C_2 \vee \cdots \vee C_n,$$

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$$u = C_1 \vee C_2 \vee \cdots \vee C_n,$$
where $C_1, ..., C_n$ are configurations.
- An instruction p acts on a single configuration of an ID u to create a new configuration u'.

Computations

We view computations as order relations on the free commutative idempotent semiring $\mathbf{A}_M = (A_M, \vee, \cdot, \perp, 1)$ generated by $Q \cup R_k$, where $M = (R_k, Q, P)$ is a k-ACM and

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Each instruction $p \in P$ defines a relation $\leq^p$ closed under

$$\frac{u \leq^p v}{ux \leq^p vx}\ [\cdot] \qquad \text{and} \qquad \frac{u \leq^p v}{u \vee w \leq^p v \vee w}\ [\vee],$$

for $u, v, w \in \mathrm{ID}(M)$ and $x \in R_k^*$, where R_k^* is the free commutative monoid generated by R_k.

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We define the **computation relation** $\leq_M$ to be the smallest preorder containing $\bigcup_{p \in P} \leq^p$.

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- If $u \leq_M q_f$, then there exists $p_1, ..., p_n \in P$ and $u_0, ..., u_n \in \mathrm{ID}(M)$, such that
$$u = u_0 \leq^{p_1} u_1 \leq^{p_2} \cdots \leq^{p_n} u_n = q_f.$$

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Example Machine

Let $M = M_{\text{even}} := (\{r\}, \{q_0, q_1, q_f\}, \{p_1, p_2, p_3\})$, with instructions

$$q_0 r \leq^{p_1} q_1; \quad q_1 r \leq^{p_2} q_0; \quad q_0 \leq^{p_3} q_f.$$

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Undecidable Problem

Theorem [Lincoln *et. al.*, 1992]

There exists a 2-ACM $\widetilde{M}$ such that membership of the set $\{u \in \mathrm{ID}(\widetilde{M}) : u \leq_{\widetilde{M}} q_f\}$ is undecidable. Furthermore, it is undecidable whether $q_I \leq_{\widetilde{M}} q_f$.

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- Given an ACM M we define the **theory of** M $\mathrm{Th}(M)$ to be the conjunction of all syntactic instructions in P, i.e.,

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$$\mathrm{Th}(M) := \bigwith\{C \leq u : (C \leq^p u) \in P\}.$$
- Given an ID u, we define the quasi-equation $\mathrm{Halt}_M(u)$ to be
$$\mathrm{Th}(M) \implies u \leq q_f.$$

d-rules and the relation $\leq_{\mathrm{d}(M)}$

Given a d-rule, e.g. $[\mathrm{d}]$ is given by $x \leq x^2 \vee x^4$, we add "ambient" instructions of the form

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for each $q \in Q$ and any $x, y \in R_k^*$.
As with the instructions in P, we close $\leq^{\mathrm{d}}$ under the inference rules $[\cdot]$ and $[\vee]$, and we define the relation $\leq_{\mathrm{d}(M)}$ to be the smallest preorder generated by $\leq^{\mathrm{d}} \cup \leq_M$.

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- However, for some ACM's M, it's possible that $u \leq_{\mathrm{d}(M)} q_f$ but $u \not\leq_M q_f$.

Example

Consider $M = M_{\mathrm{even}}$ and (d) given by $x \leq x^2 \vee x^4$.

- Clearly, if $u \leq_M q_f$ then $u \leq_{\mathrm{d}(M)} q_f$ since $\leq_M \subset \leq_{\mathrm{d}(M)}$.
- However, for some ACM's M, it's possible that $u \leq_{\mathrm{d}(M)} q_f$ but $u \not\leq_M q_f$.

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- However, $q_0 r^3 \leq_{\mathrm{d}(M)} q_f$, witnessed by

$$q_0 r^3 = q_0 r^2 r \leq^{\mathrm{d}} q_0 r^2 r^2 \vee q_0 r^2 r^4 = q_0 r^4 \vee q_0 r^6 \leq_{\mathrm{d}(M)} q_f,$$

since $q_0 r^4 \leq_M q_f$ and $q_0 r^6 \leq_M q_f$.

Goal

Given an ACM M and a d-rule, is it possible to construct a new ACM M' such that

$$u \leq_M q_f \text{ if and only if } \theta(u) \leq_{\mathrm{d}(M')} q_F,$$

(where $\theta : \mathrm{ID}(M) \to \mathrm{ID}(M')$ is computable and q_F is the final state of M') and if so, under what conditions?

Then M_K machine

Let $M = (R_2, Q, P)$ be a 2-ACM and let $K > 1$ be given. We define the 3-ACM $M_K = (R_3, Q_K, P_K)$ such that

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- each increment and decrement instruction of P is replaced by multiply and divide by K **programs**, i.e.

$$\begin{array}{llllll} q & \leq^p & q'r & \in P & \Longrightarrow & qr^{\forall} \sqsubseteq^p q'r^{K\cdot\forall} & \subset P_K \\ qr & \leq^p & q' & \in P & \Longrightarrow & qr^{\forall} \sqsubseteq^p q'r^{K\backslash\forall} & \subset P_K \end{array}.$$

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- We obtain, for each $q \in Q$,

$$qr_1^{n_1} r_2^{n_2} \leq_M q_f \iff qr_1^{K^{n_1}} r_2^{K^{n_2}} \leq_{M_K} q_F.$$

Detecting applications of $\leq^{\mathrm{d}}$

Observation

Consider a configuration where the contents of some register r is $n = s + t$, whereafter $\leq^{\mathrm{d}}$ is applied to t-many tokens, i.e.,

$$qr^n = qr^s r^t \leq^{\mathrm{d}} qr^s(r^{2t} \vee r^{4t}) = qr^{s+2t} \vee qr^{s+4t}$$

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Fact

For $\mathrm{d}: \; x \leq x^2 \vee x^4$, if $K \geq (4-2)+1 = 3$, it is **impossible** for $s + 2t$ and $s + 4t$ to **both be powers of** K.

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- Such a K will exist for any rule satisfying $(\star)$.
- Consequently, $qr^n \leq_{\mathrm{d}(M_K)} q_f$ iff $qr^n \leq_{M_K} q_F$.
- For rules in more than one variable, satisfying $(\star\star)$ is sufficient to guarantee “detection.”

Let $M = \widetilde{M} = (R_2, Q, P)$ be the 2-ACM such that it is undecidable whether $q_I \leq_M q_f$. Consider the rule (d) be given by $x \leq x^2 \vee x^4$. We construct $M_K = (R_3, Q_K, P_K)$ for $K = 3$.

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We construct $M_K = (R_3, Q_K, P_K)$ for $K = 3$.
By the observation, for any $q' \in Q_3$,

$$q' r_1^{n_1} r_2^{n_2} r_3^{n_3} \leq_{M_3} q_F \iff q' r_1^{n_1} r_2^{n_2} r_3^{n_3} \leq_{\mathrm{d}(M_3)} q_F.$$

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Hence, for any $q \in Q$,

$$q r_1^{n_1} r_2^{n_2} \leq_M q_f \iff q r_1^{3^{n_1}} r_2^{3^{n_2}} \leq_{\mathrm{d}(M_3)} q_F,$$

so it is undecidable whether $q_I r_1 r_2 \leq_{\mathrm{d}(M_3)} q_F$.

Undecidable word problem

Let $\mathcal{V} \subseteq \mathcal{CRL}$ be a variety. We can show $\mathcal{V}$ has an undecidable word problem (and hence quasi-equational theory) if we can demonstrate

$$\mathcal{V} \models \mathrm{Halt}_{\mathrm{d}(M_K)}(q_I r_1 r_2) \iff q_I r_1 r_2 \leq_M q_f.$$

- If $\mathcal{V} \subseteq \mathcal{CRL}$ then $(\Leftarrow)$ is immediate.
- We use the theory of **Residuated Frames** (Galatos & Jipsen 2013) for a completeness of encoding to provide a model and valuation proving the contrapositive of $(\Rightarrow)$, for varieties $\mathcal{V}$ satisfying certain conditions.

Residuated frames

Definition [Galatos & Jipsen 2013]

A **residuated frame** is a structure $\mathbf{W} = (W, W', N, \circ, \mathbin{\backslash\!\!\backslash}, /\!\!/, 1)$, s.t.

- $(W, \circ, 1)$ is a monoid and W' is a set.
- $N \subseteq W \times W'$, called the *Galois relation*, and
- $\mathbin{\backslash\!\!\backslash} : W \times W' \to W'$ and $/\!\!/ : W' \times W \to W'$ such that
- N is a **nuclear**, i.e. for all $u, v \in W$ and $w \in W'$,
  $(u \circ v) \; N \; w$ iff $u \; N \; (w \mathbin{/\!\!/} v)$ iff $v \; N \; (u \mathbin{\backslash\!\!\backslash} w)$.

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Define $^{\triangleright} : \mathcal{P}(W) \to \mathcal{P}(W')$ and $^{\triangleleft} : \mathcal{P}(W') \to \mathcal{P}(W)$ via
$X^{\triangleright} = \{y \in W' : \forall x \in X,\ xNy\}$ and
$Y^{\triangleleft} = \{x \in W : \forall y \in Y,\ xNy\}$, for each $X \subseteq W$ and $Y \subseteq W'$.
Then $(^{\triangleright}, ^{\triangleleft})$ is a Galois connection.
So $X \xmapsto{\gamma_N} X^{\triangleright\triangleleft}$ is a closure operator on $\mathcal{P}(W)$.

Residuated frames cont.

Theorem [Galatos & Jipsen 2013]

$\mathbf{W}^+ := (\gamma_N[\mathcal{P}(W)], \cup_{\gamma_N}, \cap, \circ_{\gamma_N}, \backslash\!\!\backslash, /\!\!/, \gamma_N(\{1\}))$,

$$X \cup_{\gamma_N} Y = \gamma_N(X \cup Y) \text{ and } X \circ_{\gamma_N} Y = \gamma_N(X \circ Y),$$

is a residuated lattice.

Proposition [Galatos & Jipsen 2013]

All simple rules are preserved by $(-)^+$.

Termination as a nuclear relation

Let $M = (R_k, Q, P)$ be a k-ACM and $W := (Q \cup R_k)^*$ be the free commutative monoid generated by $Q \cup R_k$.

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The frame $\mathbf{W_M}$

Similar to Chvalovský & Horčík (2016) , we let $W' := W$ and define the relation $N_M \subseteq W \times W'$ via

$$x \, N_M \, z \quad \text{iff} \quad xz \leq_M q_f,$$

for all $x, z \in W$. Observe that, for any $x, y, z \in W$,

$$xy \, N_M \, z \iff xyz \leq_M q_f \iff x \, N_M \, yz.$$

Since W is commutive it follows that N_M is nuclear.

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Lemma

$\mathbf{W_M} := (W, W', N_M)$ *is a residuated frame,* $\mathbf{W}^+ \in \mathcal{CRL}$*, and there exists a valuation* $\nu : \mathrm{Fm} \to W^+$ *such that* $\mathbf{W}^+, \nu \models \mathrm{Th}(M)$.

Lemma

Let (d) *be any rule satisfying* $(\star)$*. Define* $\mathbf{W}_{\mathrm{d}(\mathbf{M})} := (W, W', N_{\mathrm{d}(M)})$*. Then* $\mathbf{W}^+_{\mathrm{d}(M)} \in \mathcal{CRL}_{\mathrm{d}}$*.*

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Fix $M = \widetilde{M}$ be the 2-ACM such that it is undecidable whether $q_I \leq_M q_f$.

Theorem

Let (d) *be a rule satisfying* $(\star)$ *and* $(\star\star)$*, and let* $K \geq 2$ *be sufficiently large. Then it is undecidable whether* $\mathbf{W}^{+}_{\mathrm{d}(M_K)} \models \mathrm{Halt}_{\widetilde{M_K}}(q_I r_1 r_2)$*.*

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Corollary

For any variety $\mathcal{V} \subseteq \mathcal{CRL}$*, if*

$$\mathbf{W}^{+}_{\mathrm{d}(M_K)} \in \mathcal{V},$$

then $\mathcal{V}$ *has an undecidable word problem, and hence an undecidable quasi-equational theory.*

Known results for Equational Theory

(k_n^m) represents the knotted rule $x^n \leq x^m$

Undecidable Eq. Theory	Decidable Eq. Theory
	$\mathcal{RL}$
	$\mathcal{CRL}$
$\mathcal{RL} + (\mathrm{k}_n^m), 1 \leq n < m$	
	$\mathcal{CRL} + (\mathrm{k}_n^m)$
$\mathcal{CRL} + (?)$	

We can encode the instructions of an ACM $M = (R_k, Q, P)$ as a single term θ_M using the full signature of of $\mathcal{CRL}$ via

$$\theta_M := 1 \wedge \bigwedge_{(C \leq_M u) \in P} C \to u.$$

Let (d) be given such that there exists $n \geq 1$ and $k, c_1, ..., c_n \geq 1$ such that

$$\mathbf{CRL}_{\mathrm{d}} \models x^k \leq \bigvee_{i=1}^{n} x^{k+c_i}, \qquad (\star\star\star)$$

then (d) can be used to "bootstrap" the undeciablity of the quasi-equation theory of $\mathcal{CRL}_{\mathrm{d}}$ to the equational theory.

Undecidable equational theory

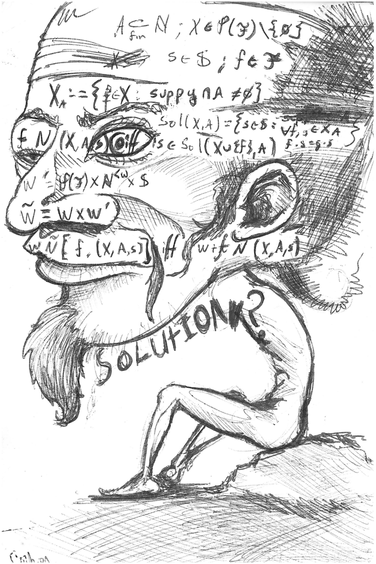
Corollary

Let (d) *be a rule satisfying* $(\star)$, $(\star\star)$, $(\star\star\star)$ *and let* $K \geq 2$ *be sufficiently large. Then it is undecidable whether*

$$\mathcal{CRL}_{\mathrm{d}} \models \theta_{M_K} \to (q_I r_1 r_2 \to q_F),$$

and therefore $\mathcal{CRL}_{\mathrm{d}}$ *has an undecidable equational theory.*

Thank You!



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