On the complexity of the equational theory of generalized residuated boolean algebras

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R-algebras

A *residuated Boolean algebra*, or *r-algebra*,(B.Jónsson and Tsinakis) is an algebra $\mathbf{A} = (\mathrm{A}, \wedge, \vee, ', \top, \bot, \cdot, \backslash, /)$ where $(\mathrm{A}, \wedge, \vee, ', \top, \bot)$ is a Boolean algebra, and $\cdot, \backslash$ and $/$ are binary operators on A satisfying the following residuation property: for any $a, b, c \in \mathrm{A}$,

$$a \cdot b \leq c \quad \text{iff} \quad b \leq a \backslash c \quad \text{iff} \quad a \leq c/b$$

The operators $\backslash$ and $/$ are called *right* and *left* residuals of $\cdot$ respectively.

The left and right conjugates of $\cdot$ are binary operators on A defined by setting

$$a \rhd c = (a \backslash c')' \text{ and } c \rhd b = (c'/b)'.$$

The following conjugation property holds for any $a, b, c \in \mathrm{A}$:

$$a \cdot b \leq c' \quad \text{iff} \quad a \rhd c \leq b' \quad \text{iff} \quad c \lhd b \leq a'$$

Let $\mathbb{K}$ be any class of algebras. The equational theory of $\mathbb{K}$, denoted by $Eq(\mathbb{K})$, is the set of all equations of the form $s = t$ that are valid in $\mathbb{K}$. The universal theory of $\mathbb{K}$ is the set of all first-order universal sentences that are valid in $\mathbb{K}$ denoted by $Ueq(\mathbb{K})$,

- $Eq(\mathbb{NA})$ is decidable (Németi 1987)
- $Eq(\mathbb{UR})$ is decidable. (Jipsen 1992)
- $Ueq(\mathbb{UR})$ and $Ueq(\mathbb{RA})$ are decidable (Buszkowski 2011)
- $Eq(\mathbb{ARA})$ is undecidable (Kurucz, Nemeti, Sain and Simon 1993)

Generalized residuated Boolean algebra

Generalized residuated algebras admit a finite number of finitary operations o. With each n-ary operation (o_i) $(1 \leq i \leq m)$ there are associated n residual operations (o_i/j) $(1 \leq j \leq n)$ which satisfy the following generalized residuation law:

$$(o_i)(\alpha_1, \ldots, \alpha_n) \leq \beta \quad \text{iff} \quad \alpha_j \leq (o_i/j)(\alpha_1, \ldots, \alpha_{j-1}, \beta, \alpha_{j+1}, \ldots, \alpha_n)$$

A generalized residuated Boolean algebra is a Boolean algebra with generalized residual operations. A generalized residuated distributive lattice and lattice are defined naturally. The logics are denoted by RBL, RDLL, RLL respectively.

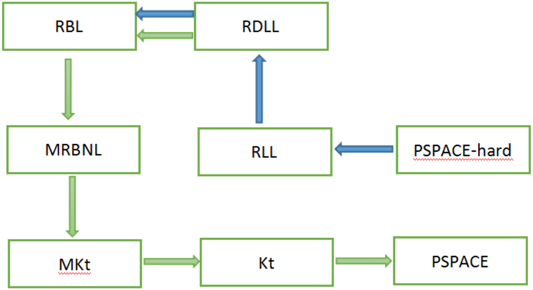

Figure: Outline of Proof

$$(\mathrm{Id})\ A \Rightarrow A, \quad (\mathrm{D})\ A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C),$$

$$(\bot)\ \Gamma[\bot] \Rightarrow A, \quad (\top)\ \Gamma \Rightarrow \top,$$

$$(\neg 1)\ A \wedge \neg A \Rightarrow \bot, \quad (\neg 2)\ \top \Rightarrow A \vee \neg A,$$

$$(\wedge \mathrm{L})\ \frac{\Gamma[A_i] \Rightarrow B}{\Gamma[A_1 \wedge A_2] \Rightarrow B}, \quad (\wedge \mathrm{R})\ \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B},$$

$$(\vee \mathrm{L})\ \frac{\Gamma[A_1] \Rightarrow B \quad \Gamma[A_2] \Rightarrow B}{\Gamma[A_1 \vee A_2] \Rightarrow B}, \quad (\vee \mathrm{R})\ \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}.$$

$$(\mathrm{Cut}) \quad \frac{\Delta \Rightarrow A; \quad \Gamma[A] \Rightarrow B}{\Gamma[\Delta] \Rightarrow B}$$

$$\frac{\Gamma[(\varphi_1, \ldots, \varphi_n)_{o_i}] \Rightarrow \alpha}{\Gamma[(o_i)(\varphi_1, \ldots, \varphi_n)] \Rightarrow \alpha}(o_i L) \quad \frac{\Gamma_1 \Rightarrow \varphi_1; \ldots; \Gamma_n \Rightarrow \varphi_n}{(\Gamma_1, \ldots, \Gamma_n)_{o_i} \Rightarrow \alpha}(o_i R)$$

$$\frac{\Gamma[\varphi_j] \Rightarrow \alpha, ; \Gamma_1 \Rightarrow \varphi_1; \ldots; \Gamma_n \Rightarrow \varphi_n}{\Gamma[(\Gamma_1, \ldots, (o_i/j)(\varphi_1, \ldots, \varphi_n), \ldots, \Gamma_n)_{o_i}] \Rightarrow \alpha}((o_i/j)L)$$

$$\frac{(\varphi_1, \ldots, \Gamma, \ldots, \varphi)_{o_i} \Rightarrow \alpha}{\Gamma \Rightarrow (o_i/j)(\varphi_1, \ldots, \Gamma, \ldots, \varphi)}((o_i/j)R)$$

Remark

All above rules are invertible.

A *frame* is a pair $\mathfrak{F} = (W, R)$ where $W \neq \emptyset$ and $R \subseteq W^{n+1}$ is an $n+1$-ary relation on W. A *model* is a triple $\mathfrak{M} = (W, R, V)$ where (W, R) is a frame and $V : \mathcal{P} \to \wp(W)$ is a valuation from the set of propositional variables $\mathcal{P}$ to the powerset of W.

The satisfaction relation $\mathfrak{M}, w \models \varphi$ between a model $\mathfrak{M}$ with a point w and a formula φ is defined inductively as follows:

1. $\mathfrak{M}, w \models p$ iff $w \in V(p)$.
2. $\mathfrak{M}, w \not\models \bot$.
3. $\mathfrak{M}, w \models \varphi \supset \psi$ iff $\mathfrak{M}, w \not\models \varphi$ or $\mathfrak{M}, w \models \psi$.
4. $\mathfrak{M}, w \models o(\varphi_1, \ldots, \varphi_n)$ iff there are points $u_1, \ldots, u_n \in W$ such that $Rwu_1 \ldots u_n$ and $\mathfrak{M}, u_i \models \varphi_i$ for $1 \leq i \leq n$.
5. $\mathfrak{M}, w \models (o/i)(\varphi_1, \ldots, \varphi_n)$ iff for all $u_1, \ldots, u_n \in W$, if $Ru_i u_1 \ldots w \ldots u_n$ and $\mathfrak{M}, u_j \models \varphi_j$ for all $1 \leq j \leq n$ and $j \neq i$, then $\mathfrak{M}, u_i \models \varphi_i$.

Unary case:

1. $\mathfrak{M}, w \models \Diamond A$ iff there exists $u \in W$ with $R(w, u)$ and $\mathfrak{M}, u \models A$.
2. $\mathfrak{M}, w \models \Box^{\downarrow} A$ iff for every $u \in W$, if $R(u, w)$, then $\mathfrak{M}, u \models A$.

Binary case:

1. $\mathfrak{J}, u \models A/B$ iff for all $v, w \in W$ with $S(w, u, v)$, if $\mathfrak{J}, v \models B$, then $\mathfrak{J}, w \models A$
2. $\mathfrak{J}, u \models A\backslash B$ iff for all $v, w \in W$ with $S(v, w, u)$, if $\mathfrak{J}, w \models A$, then $\mathfrak{J}, v \models B$.

From RBL to MRBNL

The translation $(.)^{\#} : \mathcal{L}_{\mathrm{RBL}}(\mathsf{Prop}) \to \mathcal{L}_{\mathrm{MRBNL}}(\mathsf{Prop})$ is defined as below:

- $o_i(\alpha_1, \ldots \alpha_n)^{\ddagger} = (\ldots(\alpha_1 \cdot_i \alpha_2)\ldots) \cdot_i \alpha_n)\ldots)$
- $(o_i/j)(\alpha_1, \ldots, \alpha_n) =$ $(\ldots(\alpha_1 \cdot_i \alpha_2)\ldots) \cdot_i \alpha_{j-1})\backslash_i(\ldots(\alpha_j/_i\alpha_n)\ldots/_i\alpha_{j+1})$
- $((\Gamma_1, \ldots, \Gamma_n)_{o_i})^{\ddagger} = (\ldots(\Gamma_1 \circ_i \Gamma_2)\ldots) \circ_i \Gamma_n)\ldots)$

Theorem

For any $\mathcal{L}_{\mathrm{RBL}}$-sequent $\Gamma \Rightarrow \alpha$, $\vdash_{\mathrm{RBL}} \Gamma \Rightarrow \alpha$ if and only if $\vdash_{\mathrm{MRBNL}} ((\Gamma))^{\dagger} \supset \alpha^{\dagger}$.

From MRBNL to MK_t

The translation $(.)^{\#} : \mathcal{L}_{\mathrm{MBFNL}}(\mathrm{Prop}) \to \mathcal{L}_{\mathrm{MK_t}}(\mathrm{Prop})$ is defined as below:

$$p^{\#} = p, \qquad \top^{\#} = \top, \quad \bot^{\#} = \bot,$$
$$(\neg\alpha)^{\#} = \neg\alpha^{\#}, \qquad (\alpha \wedge \beta)^{\#} = \alpha^{\#} \wedge \beta^{\#},$$
$$(\alpha \vee \beta)^{\#} = \alpha^{\#} \vee \beta^{\#}, \qquad (\alpha \cdot_i \beta)^{\#} = \Diamond_{i1}(\Diamond_{i1}\alpha^{\#} \wedge \Diamond_{i2}\beta^{\#}),$$
$$(\alpha\backslash_i\beta)^{\#} = \Box^{\downarrow}_{i2}(\Diamond_{i1}\alpha^{\#} \supset \Box^{\downarrow}_{i1}\beta^{\#}), \quad (\alpha/_i\beta)^{\#} = \Box^{\downarrow}_{i1}(\Diamond_{i2}\beta^{\#} \supset \Box^{\downarrow}_{i1}\alpha^{\#}).$$

Theorem

For any $\mathcal{L}_{\mathrm{MBFNL}}$-sequent $\Gamma \Rightarrow \alpha$, $\vdash_{\mathrm{MBFNL}} \Gamma \Rightarrow \alpha$ if and only if $\vdash_{\mathrm{MK_t}} (f(\Gamma))^{\#} \supset \alpha^{\#}$.

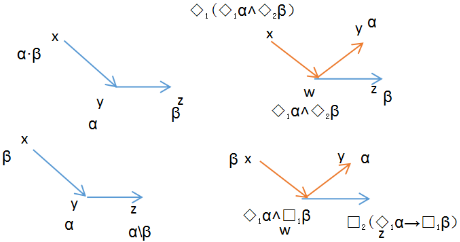

Figure: Translation #

From MK_t to K_t

Let $\mathrm{P} \subseteq \mathrm{Prop}$ and $\{x, q_1, \ldots, q_n\} \not\subseteq \mathrm{P}$ be a distinguished propositional variable. Define a translation $(.)^* : \mathcal{L}_{\mathrm{K}^{\mathrm{t}}_{12}}(\mathrm{P}) \to \mathcal{L}_{\mathrm{K.t}}(\mathrm{P} \cup \{x, q_1, \ldots, q_n\})$ recursively as follows:

$$p^* = p, \bot^* = \bot,$$

$$(A \supset B)^* = A^* \supset B^*.$$

$$(\Diamond_i A)^* = \neg x \wedge \Diamond(q_i \wedge A^*),$$

$$(\Box_i^{\downarrow} A)^* = \neg x \supset \Box^{\downarrow}(q_i \supset A^*),$$

Theorem

For any $\mathcal{L}_{\mathrm{MK_t}}$*-sequent* $\Gamma \Rightarrow \alpha$, $\vdash_{\mathrm{MK_t}} \Gamma \Rightarrow \alpha$ *if and only if* $\vdash_{\mathrm{K_t}} (f(\Gamma))^* \supset \alpha^*$.

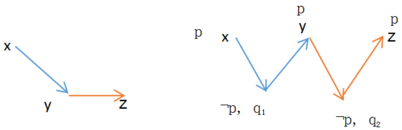

Figure: Translation $*$

$$(\mathrm{Id}) \quad A \Rightarrow A,$$

and inference rules

$$(\cdot \mathrm{L}) \quad \frac{\Gamma[A \circ B] \Rightarrow C}{\Gamma[A \cdot B] \Rightarrow C}, \quad (\cdot \mathrm{R}) \quad \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma \circ \Delta \Rightarrow A \cdot B},$$

$$(\mathrm{Cut}) \quad \frac{\Delta \Rightarrow A; \quad \Gamma[A] \Rightarrow B}{\Gamma[\Delta] \Rightarrow B}$$

$$(\wedge \mathrm{L}) \, \frac{\Gamma[A_i] \Rightarrow B}{\Gamma[A_1 \wedge A_2] \Rightarrow B}, \quad (\wedge \mathrm{R}) \, \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B},$$

$$(\vee \mathrm{L}) \, \frac{\Gamma[A_1] \Rightarrow B \quad \Gamma[A_2] \Rightarrow B}{\Gamma[A_1 \vee A_2] \Rightarrow B}, \quad (\vee \mathrm{R}) \, \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}.$$

$(\cdot$ L), $(\cdot$ R), $(\wedge R)$ and $(\vee L)$ are invertible.

PSPACE-hard

Lemma

If $\vdash_{LG} \Gamma[A \wedge B] \Rightarrow C$ *and all formulae in* $\Gamma[A \wedge B]$ *are* $\vee$*-free and* C *is* $\wedge$*-free, then* $\Gamma[A] \Rightarrow C$ *or* $\Gamma[B] \Rightarrow C$*.*

Lemma

If $\vdash_{LG} \Gamma \Rightarrow A \vee B$ *and all formulae in* Γ *are* $\vee$*-free, then* $\Gamma \Rightarrow A$ *or* $\Gamma[B] \Rightarrow B$*.*

By $\sigma(e)$ we denote a formula structure $z_1 \circ (z_2 \cdots (z_{n-1} \circ z_n) \cdots)$ such that

$$z_j = \begin{cases} x_j & \textit{if} \quad e(x_j) = 1 \\ \overline{x_j} & \textit{if} \quad e(x_j) = 0 \end{cases}$$

$\sigma(A) = \sigma(D_1) \vee \ldots \vee \sigma(D_m)$ and
$\sigma(D_i) = y_1 \cdot (y_2 \cdots (y_{n-1} \cdot y_n) \cdots)$ such that

$$y_j = \begin{cases} x_j & \textit{if} \quad x_j \in D_i \\ \overline{x_j} & \textit{if} \quad \neg x_j \in D_i \\ x_j \vee \overline{x_j} & o.w. \end{cases}$$

Lemma

$e(A) = 1$ *iff* $\vdash_{LG} \sigma(e) \Rightarrow \sigma(A)$

Let us consider a quantified Boolean formula ϕ in DNF form i.e. $\phi = Q_k x_k \cdots Q_1 x_1 A$ where $Q_i \in \{\forall, \exists\}$ and A is a propositional formulae in DNF form. We extended the translation of $e(\phi)$ into a sequent in LG as follows: $\sigma(e)$ we denote a formula structure $z_1 \circ (z_2 \cdots (z_{n-1} \circ z_n) \cdots)$ such that for any $1 \leq j \leq k$

$$z_j = \begin{cases} x_j \wedge \overline{x_j} & if \quad Q_j = \exists \\ x_j \vee \overline{x_j} & if \quad Q_j = \forall \end{cases}$$

and for any $k+1 \leq j \leq n$ z_j is defined as above. Further the translation on A is remained the same.

Theorem

$e(\phi) = 1$ iff $\sigma(e) \Rightarrow \sigma(A)$ where A is a quantifier free formula of ϕ.

Theorem

The decision problem of LG is PSPACE-hard.

We define two special sub-languages of LG and DLG. The Left sub-language of LG and DLG denoted by LL is defined recursively as follows:

$$A ::= p \mid p \wedge p \mid p \vee p \mid (A \cdot A)$$

The right sub-language of LG and DLG denoted by RL is defined recursively as follows:

$$A ::= p \mid p \vee p \mid (A \cdot A)$$

Lemma

Given a sequent $\Gamma \Rightarrow A$ such that Γ is a LL formula structure and A is a RL formula. Then $\vdash_{LG} \Gamma \Rightarrow A$ iff $\vdash_{DLG} \Gamma \Rightarrow A$.

Theorem

The decision problem of RBL, RDLL, RLL are PSPACE-hard.

Remark

By Buszkowski[2011], RBL is conservative extension of RDLL, while RDLL and RLL are conservative extension of DLG and LG respectively

PSPACE-completeness

Theorem

The decision problem of RBL, RDLL, RLL are PSPACE-complete.

Extensions

For any extensions S of RBL, RDLL, RLL with set of axioms ϕ, if $(\cdot L)$ and $(\cdot R)$ are both invertible, then the decision problem of S is PSPACE-hard.

For instance, $\mathrm{FNL_e}$, $\mathrm{FNL_c}$, $\mathrm{DFNL_e}$, ...

For any extensions S of RLL with set of axioms ϕ, if $(\cdot L)$ and $(\cdot R)$ re both invertible and admit cut elimination, then the decision problem of S is PSPACE-complete.

For instance, $\mathrm{FNL_e}$, ...

Thank you