Deepening the link between logic and functional analysis via Riesz MV-algebras

Serafina Lapenta

includes joint works with Antonio Di Nola and Ioana Leuştean

TACL 2017, Prague

26-30 June 2017

MV-algebras

MV-algebras

- Algebraic counterpart of Łukasiewicz logic,

MV-algebras

- Algebraic counterpart of Łukasiewicz logic,
- A generalization of boolean algebras,

MV-algebras

- Algebraic counterpart of Łukasiewicz logic,
- A generalization of boolean algebras,
- Unit interval of lattice ordered groups with strong unit.

Unit interval of lattice ordered groups with strong unit

Unit interval of lattice ordered groups with strong unit

Lattice-ordered groups

$(G, +, 0, \leq)$ is *ℓ-group*

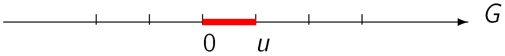
- if $(G, +, 0)$ group,
- $(G, \leq)$ lattice,
- $x \leq y$ implies $x + z \leq y + z$ for any $x, y, z \in G$.

ℓu-groups

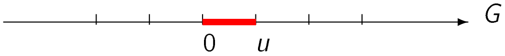
$u \in G$ is a **strong unit**: $u \geq 0$, for any $x \in G$ there is $n \geq 1$ s.t. $x \leq nu$.

An abelian ℓ-group with strong unit is an ℓu-group.

Mundici's categorical equivalence

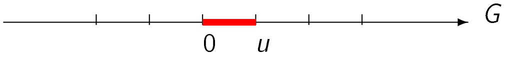


Mundici's categorical equivalence



(G, u) ℓu-group, $x, y \in G$, $x \oplus y = (x +_G y) \wedge u$, $\quad x^* = u -_G x$

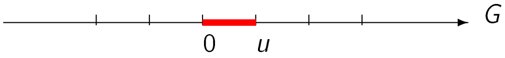
Mundici's categorical equivalence



(G, u) ℓu-group, $x, y \in G$, $x \oplus y = (x +_G y) \wedge u$, $\quad x^* = u -_G x$

$([0, u]_G, \oplus, ^*, 0)$ is an MV-algebra

Mundici's categorical equivalence



(G, u) ℓu-group, $x, y \in G$, $x \oplus y = (x +_G y) \wedge u$, $\quad x^* = u -_G x$

$([0, u]_G, \oplus, ^*, 0)$ is an MV-algebra

Categorical equivalence. Mundici, 1986

$\Gamma(G, u) = [0, u]_G$,

$f : (G, u) \to (H, v) \mapsto f|_{[0,u]_G} : [0, u]_G \to [0, v]_H$

MV-algebras with product: Riesz MV-algebras

MV-algebras R endowed with a scalar multiplication with scalars from $[0, 1]$.

A. Di Nola, I. Leuștean, 2014

MV-algebras with product: Riesz MV-algebras

MV-algebras R endowed with a scalar multiplication with scalars from $[0,1]$.

A. Di Nola, I. Leuştean, 2014

- Standard model: $[0,1]_{RMV} = [0,1]_{MV}$ + scalar multiplication from $[0,1]$,
- RMVs form a variety,
- $\mathbb{RMV} = HSP([0,1]_{RMV})$,
- as a category, are equivalent to Riesz Spaces (vector lattices) with a strong unit.

Logics

We can expand Łukasiewicz logic with connectives that model the scalar multiplication and obtain a new logic, $\mathbb{R}\mathcal{L}$.

Logics

We can expand Łukasiewicz logic with connectives that model the scalar multiplication and obtain a new logic, $\mathbb{R}\mathcal{L}$.

$\mathbb{R}\mathcal{L}$ is a conservative extension of $\mathcal{L}$

Boolean algebras	$\Leftrightarrow$	Classical logic
MV-algebras	$\Leftrightarrow$	Łukasiewicz logic $\mathcal{L}$
Riesz MV-algebras	$\Leftrightarrow$	Riesz Logic $\mathbb{R}\mathcal{L}$

Functional representations

Functional representations

$f : [0,1]^n \to [0,1]$ is a $\mathrm{PWL}_u(\mathbb{Z})$ function if it is continuous and there is a finite set of affine functions $p_1, \ldots, p_k : \mathbb{R}^n \to \mathbb{R}$ with integer coefficients such that for any $(a_1, \ldots, a_n) \in [0,1]^n$ there exists $i \in \{1, \ldots, k\}$ with $f(a_1, \ldots, a_n) = p_i(a_1, \ldots, a_n)$.

Functional representations

$f : [0,1]^n \to [0,1]$ is a PWL$_u(\mathbb{Z})$ function if it is continuous and there is a finite set of affine functions $p_1, \ldots, p_k : \mathbb{R}^n \to \mathbb{R}$ with integer coefficients such that for any $(a_1, \ldots, a_n) \in [0,1]^n$ there exists $i \in \{1, \ldots, k\}$ with $f(a_1, \ldots, a_n) = p_i(a_1, \ldots, a_n)$.

Free MV-algebra $MV_n \simeq Lind_{\mathcal{L},n}$ [R. McNaughton, 1951]

$MV_n = \{f_\varphi : [0,1]^n \to [0,1] \mid \varphi \text{ formula of } \mathcal{L}\}$=PWL$_u(\mathbb{Z})$

Functional representations

Functional representations

$f : [0,1]^n \to [0,1]$ is a $\mathrm{PWL}_u(\mathbb{R})$ function if it is continuous and there is a finite set of affine functions $p_1, \ldots, p_k : \mathbb{R}^n \to \mathbb{R}$ with real coefficients such that for any $(a_1, \ldots, a_n) \in [0,1]^n$ there exists $i \in \{1, \ldots, k\}$ with $f(a_1, \ldots, a_n) = p_i(a_1, \ldots, a_n)$.

Functional representations

$f : [0,1]^n \to [0,1]$ is a PWL$_u(\mathbb{R})$ function if it is continuous and there is a finite set of affine functions $p_1, \ldots, p_k : \mathbb{R}^n \to \mathbb{R}$ with real coefficients such that for any $(a_1, \ldots, a_n) \in [0,1]^n$ there exists $i \in \{1, \ldots, k\}$ with $f(a_1, \ldots, a_n) = p_i(a_1, \ldots, a_n)$.

Free Riesz MV-algebra $RMV_n \simeq Lind_{\mathbb{R}\mathcal{L},n}$ [A. Di Nola, I. Leuștean 2014]

$RMV_n = \{f_\varphi : [0,1]^n \to [0,1] \mid \varphi \text{ formula of } \mathbb{R}\mathcal{L}\}$=PWL$_u(\mathbb{R})$

Convergence in $\mathbb{R}\mathcal{L}$

Convergence in $\mathbb{R}\mathcal{L}$

For $r \in [0,1]$ we set $\mathbf{r} := \Delta_r(\theta)$ where $\vdash \theta$

Convergence in $\mathbb{R}\mathcal{L}$

For $r \in [0, 1]$ we set $\mathbf{r} := \Delta_r(\theta)$ where $\vdash \theta$

Uniform Limit of formulas

A formula φ is the uniform limit of the sequence $(\varphi_n)_n$ in $\mathbb{R}\mathcal{L}$ if for any $r < 1$ there is k such that for any $n \geq k$: $\vdash \mathbf{r} \rightarrow (\varphi \leftrightarrow \varphi_n)$. We write $\lim_n \varphi_n = \varphi$.

Convergence in $\mathbb{RL}$

For $r \in [0,1]$ we set $\mathbf{r} := \Delta_r(\theta)$ where $\vdash \theta$

Uniform Limit of formulas

A formula φ is the uniform limit of the sequence $(\varphi_n)_n$ in $\mathbb{RL}$ if for any $r < 1$ there is k such that for any $n \geq k$: $\vdash \mathbf{r} \to (\varphi \leftrightarrow \varphi_n)$. We write $\lim_n \varphi_n = \varphi$.

TFAE:

- $\lim_n \varphi_n = \varphi$,
- $\lim_n f_{\varphi_n} = f_\varphi$ (uniform convergence),
- there exists $\{f_{\psi_n}\}_{n\in\mathbb{N}}$ such that $\bigwedge_n f_{\psi_n}(x) = 0$ for all $x \in [0,1]^n$ and $d(f_{\varphi_n}, f_\varphi)(x) \leq f_{\psi_n}(x)$ in $Lind_{\mathbb{RL}}$ (strong order convergence)

A remark

A remark

- Riesz MV-algebras are equivalent with Riesz Spaces with a strong unit,

A remark

- Riesz MV-algebras are equivalent with Riesz Spaces with a strong unit,
- In a Riesz space we have three notions of convergence: uniform, in order and in norm,

A remark

- Riesz MV-algebras are equivalent with Riesz Spaces with a strong unit,
- In a Riesz space we have three notions of convergence: uniform, in order and in norm,
- Usually, order converge does not imply uniform convergence nor norm-convergence
- In $C([0,1]^n)$ with the requirement of point-wise inf, we can prove that our three notions coincide!

Thus, what about (uniform) norm-convergence and norm-completions?

Norm of formulas: the unit-norm

Norm of formulas: the unit-norm

$[\varphi]$ in $Lind_{\mathbb{R}\mathcal{L},n}$,

$$\|[\varphi]\|_u = \sup\{f_\varphi(\mathbf{x}) | \mathbf{x} \in [0,1]^n\}$$

Norm of formulas: the unit-norm

$[\varphi]$ in $Lind_{\mathbb{R}\mathcal{L},n}$,

$$\|[\varphi]\|_u = \sup\{f_\varphi(\mathbf{x}) | \mathbf{x} \in [0,1]^n\}$$

$(Lind_{\mathbb{R}\mathcal{L},n}, \|\cdot\|_u)$ is a normed space.

Norm of formulas: the unit-norm

$[\varphi]$ in $Lind_{\mathbb{R}\mathcal{L},n}$,

$$\|[\varphi]\|_u = \sup\{f_\varphi(\mathbf{x}) | \mathbf{x} \in [0,1]^n\}$$

$(Lind_{\mathbb{R}\mathcal{L},n}, \|\cdot\|_u)$ is a normed space.

Completions

The norm-completion of the normed space $(Lind_{\mathbb{R}\mathcal{L},n}, \|\cdot\|_u)$ is isometrically isomorphic with $(C([0,1]^n), \|\cdot\|_\infty)$.

Norm of formulas: the unit-norm

$[\varphi]$ in $Lind_{\mathbb{R}\mathcal{L},n}$,

$$\|[\varphi]\|_u = \sup\{f_\varphi(\mathbf{x}) | \mathbf{x} \in [0,1]^n\}$$

$(Lind_{\mathbb{R}\mathcal{L},n}, \|\cdot\|_u)$ is a normed space.

Completions

The norm-completion of the normed space $(Lind_{\mathbb{R}\mathcal{L},n}, \|\cdot\|_u)$ is isometrically isomorphic with $(C([0,1]^n), \|\cdot\|_\infty)$.

Corollary: approximation of continuous functions

For any continuous function $f : [0,1]^n \to [0,1]$ there exists a sequence of formulas $(\varphi_n)_n$ of $\mathbb{R}\mathcal{L}$ such that $\lim_n f_{\varphi_n} = f$.

Norm of formulas: the integral norm

Norm of formulas: the integral norm

$[\varphi]$ in $Lind_{\mathbb{R}\mathcal{L},n}$,

$$I([\varphi]) = \int f_{\varphi}(\mathbf{x}) d\mathbf{x}$$

Norm of formulas: the integral norm

$[\varphi]$ in $Lind_{\mathbb{RL},n}$,

$$I([\varphi]) = \int f_{\varphi}(\mathbf{x})d\mathbf{x}$$

$(Lind_{\mathbb{RL},n}, I)$ is a normed space.

Completions

The norm-completion of the normed space $(Lind_{\mathbb{RL},n}, I)$ is isometrically isomorphic with $(L^1(\mu)_u, s_\mu)$,

Norm of formulas: the integral norm

$[\varphi]$ in $Lind_{\mathbb{R}\mathcal{L},n}$,

$$I([\varphi]) = \int f_{\varphi}(\mathbf{x})d\mathbf{x}$$

$(Lind_{\mathbb{R}\mathcal{L},n}, I)$ is a normed space.

Completions

The norm-completion of the normed space $(Lind_{\mathbb{R}\mathcal{L},n}, I)$ is isometrically isomorphic with $(L^1(\mu)_u, s_\mu)$,where:

- μ be the Lebesgue measure associated to I,
- $L^1(\mu)_u$ is the algebra of $[0,1]$-valued integrable functions on $[0,1]^n$,
- $s_\mu(\hat{f}) = I(f)$ and $\hat{f}$ is the class of f, provided we identify two functions that are equal μ-almost everywhere.

Thus, we have two norm-completions of $Lind_{\mathbb{R}\mathcal{L},n}$ and one of them is $C([0,1]^n)$...

Thus, we have two norm-completions of $Lind_{\mathbb{R}\mathcal{L},n}$ and one of them is $C([0,1]^n)$...

How can we generalize this result? How can we provide a system that has $C(X)$, with X compact Hausdorff space, as models?

Disclaimer: the following is still a work in progress!!

Disclaimer: the following is still a work in progress!!

Di Nola A., Lapenta S., Leuştean I., *An analysis of the logic of Riesz Spaces with strong unit*, submitted.

Di Nola A., Lapenta S., Leuştean I., *Infinitary Riesz Logic*, in preparation.

How to get compact Hausdorff spaces from Riesz MV-algebras?

How to get compact Hausdorff spaces from Riesz MV-algebras?

- $R \in \mathbf{RMV}$, $\|\cdot\|_u : R \to [0,1]$

$$\|x\|_u = \min\{r \in [0,1] \mid x \leq r1\}$$

How to get compact Hausdorff spaces from Riesz MV-algebras?

- $R \in \mathbf{RMV}$, $\|\cdot\|_u : R \to [0,1]$

$$\|x\|_u = \min\{r \in [0,1] \mid x \leq r1\}$$

On $Lind_{\mathbb{R}\mathcal{L},n}$, it collapses on sup-norm seen before!!

How to get compact Hausdorff spaces from Riesz MV-algebras?

- $R \in \mathbf{RMV}$, $\|\cdot\|_u : R \to [0,1]$

$$\|x\|_u = \min\{r \in [0,1] \mid x \leq r1\}$$

On $Lind_{\mathbb{R}\mathcal{L},n}$, it collapses on sup-norm seen before!!

- A Riesz MV-algebra is norm-complete if it is a complete normed space wrt to $\|\cdot\|_u$.
- The unit-norm can be defined on Riesz Spaces. A norm complete Riesz space wrt the unit-norm is called M-space.

How to get compact Hausdorff spaces from Riesz MV-algebras?

- $R \in \mathbf{RMV}$, $\|\cdot\|_u : R \to [0,1]$

$$\|x\|_u = \min\{r \in [0,1] \mid x \leq r1\}$$

On $Lind_{\mathbb{R}\mathcal{L},n}$, it collapses on sup-norm seen before!!

- A Riesz MV-algebra is norm-complete if it is a complete normed space wrt to $\|\cdot\|_u$.
- The unit-norm can be defined on Riesz Spaces. A norm complete Riesz space wrt the unit-norm is called M-space.

M-spaces

An M-space is a Banach lattice (norm-complete Riesz Space) endowed with a norm ϱ such that $\varrho(x \vee y) = \max(\varrho(x), \varrho(y))$.

Kakutani's duality

The category of M-spaces and suitable morphisms is dual to the category of compact Hausdorff spaces and continuous maps.

Kakutani's duality

The category of M-spaces and suitable morphisms is dual to the category of compact Hausdorff spaces and continuous maps.

M-spaces and Riesz MV-algebras [A. Di Nola and I. Leuştean, 2014]

The category of M-spaces and suitable morphisms is equivalent to the full subcategory of norm-complete Riesz MV-algebras.

Kakutani's duality

The category of M-spaces and suitable morphisms is dual to the category of compact Hausdorff spaces and continuous maps.

M-spaces and Riesz MV-algebras [A. Di Nola and I. Leuștean, 2014]

The category of M-spaces and suitable morphisms is equivalent to the full subcategory of norm-complete Riesz MV-algebras.

Hence, Norm-complete Riesz MV-algebras are dual to compact Hausdorff spaces.

The idea

- In $Lind_{\mathbb{R}\mathcal{L},n}$, unit-norm convergence $\Leftrightarrow$ uniform convergence $\Leftrightarrow$ strong order-convergence,

The idea

- In $Lind_{\mathbb{R}\mathcal{L},n}$, unit-norm convergence $\Leftrightarrow$ uniform convergence $\Leftrightarrow$ strong order-convergence,
- $C([0,1]^n)$ is the norm-completion of $Lind_{\mathbb{R}\mathcal{L},n}$,

The idea

- In $Lind_{\mathbb{RL},n}$, unit-norm convergence $\Leftrightarrow$ uniform convergence $\Leftrightarrow$ strong order-convergence,
- $C([0,1]^n)$ is the norm-completion of $Lind_{\mathbb{RL},n}$,
- Why not to "close" $\mathbb{RL}$ for countable inf and sup of formulas?

The logic $\mathcal{IRL}$

The logic $\mathcal{IRL}$

- Language: $\{\rightarrow, \neg\} \cup \{\nabla_r\}_{r\in[0,1]} \cup \bigvee$

- Axioms: the ones of $\mathbb{R}\mathcal{L}$ +

(S1) $\varphi_k \rightarrow \bigvee_{n\in\mathbb{N}} \varphi_n$, for any $k \in \mathbb{N}$

- Deduction rules: Modus Ponens +

$$\text{(SUP)}\quad \frac{(\varphi_1 \rightarrow \psi), \ldots, (\varphi_k \rightarrow \psi) \ldots}{\bigvee_{n\in\mathbb{N}} \varphi_n \rightarrow \psi}$$

The logic $\mathcal{IRL}$

- Language: $\{\rightarrow, \neg\} \cup \{\nabla_r\}_{r\in[0,1]} \cup \bigvee$

- Axioms: the ones of $\mathbb{R}\mathcal{L}$ +

(S1) $\varphi_k \rightarrow \bigvee_{n\in\mathbb{N}} \varphi_n$, for any $k \in \mathbb{N}$

- Deduction rules: Modus Ponens +

(SUP) $$\frac{(\varphi_1 \rightarrow \psi), \ldots, (\varphi_k \rightarrow \psi) \ldots}{\bigvee_{n\in\mathbb{N}} \varphi_n \rightarrow \psi}$$

Karp C. R., *Languages with expressions of infinite length*, North-Holland Pub. Co., 1964.

(For the boolean case)

Models of the logic

Models of the logic

- Models are Dedekind σ-complete Riesz MV-algebras, and $\mathbf{RMV_{dc\sigma}}$ is their subcategory;
- $\mathcal{IRL}$ is complete wrt to all objects in $\mathbf{RMV_{dc\sigma}}$, i.e. $\vdash_{\mathcal{IRL}} \varphi$ iff $e(\varphi) = 1$ for any R-valued evaluation, with $R \in \mathbf{RMV_{dc\sigma}}$;

Models of the logic

- Models are Dedekind σ-complete Riesz MV-algebras, and $\mathbf{RMV_{dc\sigma}}$ is their subcategory;
- $\mathcal{IRL}$ is complete wrt to all objects in $\mathbf{RMV_{dc\sigma}}$, i.e. $\vdash_{\mathcal{IRL}} \varphi$ iff $e(\varphi) = 1$ for any R-valued evaluation, with $R \in \mathbf{RMV_{dc\sigma}}$;
- $Lind_{\mathcal{IRL}}$ is the Dedekind σ-completion of $Lind_{\mathbb{R}\mathcal{L}}$;

Models of the logic

- Models are Dedekind σ-complete Riesz MV-algebras, and $\mathbf{RMV_{dc\sigma}}$ is their subcategory;
- $\mathcal{IRL}$ is complete wrt to all objects in $\mathbf{RMV_{dc\sigma}}$, i.e. $\vdash_{\mathcal{IRL}} \varphi$ iff $e(\varphi) = 1$ for any R-valued evaluation, with $R \in \mathbf{RMV_{dc\sigma}}$;
- $Lind_{\mathcal{IRL}}$ is the Dedekind σ-completion of $Lind_{\mathbb{R}\mathcal{L}}$;
- Any Dedekind σ-complete Riesz MV-algebra is norm-complete.

Thus, our models are $C(X)$, for some peculiar X compact Hausdorff space:

$C(X)$ is Dedekind σ-complete iff X is quasi-Stonean (basically disconnected) [Mundici, 2011, for MV-algebras]

A corollary:

$Lind_{\mathcal{IRL}}$ is a σ-complete Riesz MV-algebra, and therefore it is also norm-complete

Thus, our models are $C(X)$, for some peculiar X compact Hausdorff space:

$C(X)$ is Dedekind σ-complete iff X is quasi-Stonean (basically disconnected) [Mundici, 2011, for MV-algebras]

A corollary:

$Lind_{\mathcal{IRL}}$ is a σ-complete Riesz MV-algebra, and therefore it is also norm-complete

$\Rightarrow$ $Lind_{\mathcal{IRL}}$ is the unit interval of a Dedekind σ-complete M-space

Thus, our models are $C(X)$, for some peculiar X compact Hausdorff space:

$C(X)$ is Dedekind σ-complete iff X is quasi-Stonean (basically disconnected) [Mundici, 2011, for MV-algebras]

A corollary:

$Lind_{\mathcal{IRL}}$ is a σ-complete Riesz MV-algebra, and therefore it is also norm-complete

$\Rightarrow$ $Lind_{\mathcal{IRL}}$ is the unit interval of a Dedekind σ-complete M-space

$\Rightarrow$ there exists a basically disconnected space X such that $Lind_{\mathcal{IRL}} \simeq C(X)$

Thus, our models are $C(X)$, for some peculiar X compact Hausdorff space:

$C(X)$ is Dedekind σ-complete iff X is quasi-Stonean (basically disconnected) [Mundici, 2011, for MV-algebras]

A corollary:

$Lind_{\mathcal{IRL}}$ is a σ-complete Riesz MV-algebra, and therefore it is also norm-complete

$\Rightarrow$ $Lind_{\mathcal{IRL}}$ is the unit interval of a Dedekind σ-complete M-space

$\Rightarrow$ there exists a basically disconnected space X such that $Lind_{\mathcal{IRL}} \simeq C(X)$

We have a functional representation of $Lind_{\mathcal{IRL}}$!

Summing up...

- We provided three different extensions of $Lind_{\mathbb{R}\mathcal{L},n}$, and

Summing up...

- We provided three different extensions of $Lind_{\mathbb{R}\mathcal{L},n}$, and

$$Lind_{\mathbb{R}\mathcal{L},n} \subseteq C([0,1]^n)$$

(i) $C([0,1]^n)$ is the norm completion of $Lind_{\mathbb{R}\mathcal{L},n}$ and it is not is not Dedekind σ-complete, thus $Lind_{\mathbb{R}\mathcal{L},n} \subseteq C([0,1]^n)$,

Summing up...

- We provided three different extensions of $Lind_{\mathbb{RL},n}$, and

$$Lind_{\mathbb{RL},n} \subsetneq C([0,1]^n) \subsetneq Lind_{\mathcal{IRL},n}$$

(i) $C([0,1]^n)$ is the norm completion of $Lind_{\mathbb{RL},n}$ and it is not is not Dedekind σ-complete, thus $Lind_{\mathbb{RL},n} \subsetneq C([0,1]^n)$,

(ii) $Lind_{\mathcal{IRL},n}$ is a Dedekind σ-complete and it is the completion of $Lind_{\mathbb{RL},n}$, thus $Lind_{\mathbb{RL},n} \subsetneq Lind_{\mathcal{IRL},n}$,

(iii) $Lind_{\mathcal{IRL}}$ is norm-complete and it contains $Lind_{\mathbb{RL},n}$, thus $C([0,1]^n) \subsetneq Lind_{\mathcal{IRL},n}$,

Summing up...

- We provided three different extensions of $Lind_{\mathbb{R}\mathcal{L},n}$, and

$$Lind_{\mathbb{R}\mathcal{L},n} \subseteq C([0,1]^n) \subseteq Lind_{\mathcal{IRL},n} \subseteq L^1(\mu)_u$$

(i) $C([0,1]^n)$ is the norm completion of $Lind_{\mathbb{R}\mathcal{L},n}$ and it is not is not Dedekind σ-complete, thus $Lind_{\mathbb{R}\mathcal{L},n} \subseteq C([0,1]^n)$,

(ii) $Lind_{\mathcal{IRL},n}$ is a Dedekind σ-complete and it is the completion of $Lind_{\mathbb{R}\mathcal{L},n}$, thus $Lind_{\mathbb{R}\mathcal{L},n} \subseteq Lind_{\mathcal{IRL},n}$,

(iii) $Lind_{\mathcal{IRL}}$ is norm-complete and it contains $Lind_{\mathbb{R}\mathcal{L},n}$, thus $C([0,1]^n) \subseteq Lind_{\mathcal{IRL},n}$,

(iv) $L^1(\mu)_u$ is another norm-completion of $Lind_{\mathbb{R}\mathcal{L},n}$ and it is Dedekind complete, thus $Lind_{\mathcal{IRL},n} \subseteq L^1(\mu)_u$.

Another approach to the same problem [Marra V. and Reggio L., 2016]

Another approach to the same problem [Marra V. and Reggio L., 2016]

- They introduce δ-algebra: MV-algebra + an infinitary operator δ such that, for sequences in $C(X)$, is $\sum_n \frac{f_n}{2^n}$.

Another approach to the same problem [Marra V. and Reggio L., 2016]

- They introduce δ-algebra: MV-algebra + an infinitary operator δ such that, for sequences in $C(X)$, is $\sum_n \frac{f_n}{2^n}$.
- They prove that **KHausd** is dual to the category of δ-algebras with δ-preserving MV-algebra morphisms.

Another approach to the same problem [Marra V. and Reggio L., 2016]

- They introduce δ-algebra: MV-algebra + an infinitary operator δ such that, for sequences in $C(X)$, is $\sum_n \frac{f_n}{2^n}$.
- They prove that **KHausd** is dual to the category of δ-algebras with δ-preserving MV-algebra morphisms.
- Hence, δ-algebras are equivalent to norm-complete Riesz MV-algebras,

Another approach to the same problem [Marra V. and Reggio L., 2016]

- They introduce δ-algebra: MV-algebra + an infinitary operator δ such that, for sequences in $C(X)$, is $\sum_n \frac{f_n}{2^n}$.
- They prove that **KHausd** is dual to the category of δ-algebras with δ-preserving MV-algebra morphisms.
- Hence, δ-algebras are equivalent to norm-complete Riesz MV-algebras,
- In our language: given $\{\varphi_n\}_n$ we define the sequence $\sigma_1 = \Delta_{\frac{1}{2}}\varphi_1$, $\sigma_2 = \Delta_{\frac{1}{2}}\varphi_1 \oplus \Delta_{\frac{1}{2^2}}\varphi_2, \ldots$ and we set $\delta(\varphi_1, \varphi_2, \cdots) = lim_n \sigma_n$.

Work in progress

- Chain completeness for $\mathcal{IRL}$
- We aim at a more concrete functional representation (with functions in $[0,1]^{[0,1]^n}$) of $Lind_{\mathcal{IRL}}$ and we are following three main ideas: MV-tribes, normal functions and Baire functions.
- We are working the extension of $\mathbb{R}\mathcal{L}$ with a deduction via limits and the impact of this ideas on the relations between $\mathbb{R}\mathcal{L}$ and the Rational Łukasiewicz logic.

Thank you!