Theories of relational lattices. AKA: Embeddability into relational lattices is undecidable[1]

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Relational lattices

Quasiequational theories of relational lattices

The lattice of a frame

p-morphisms from lattice embeddings

More on equational theory

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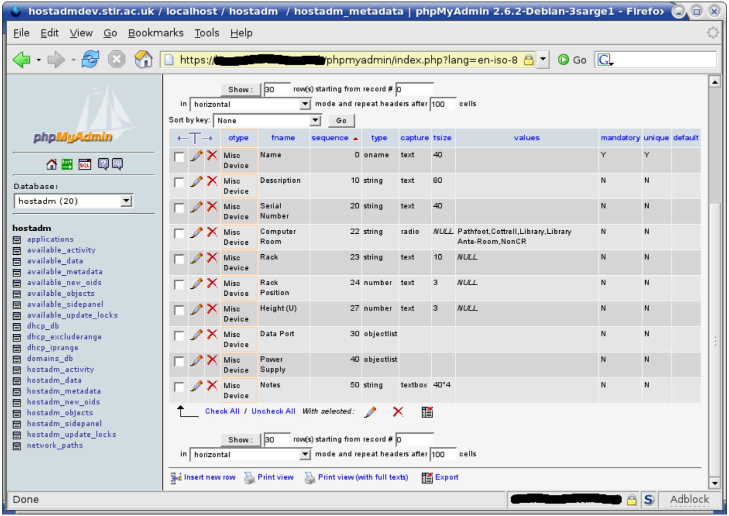
Databases, tables, sqls . . .



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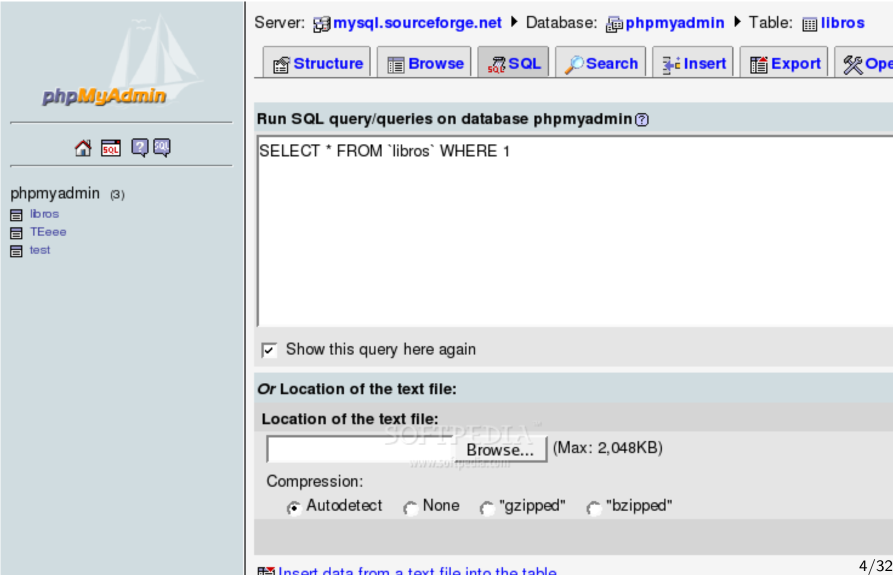

Operations on tables: the natural join

Name	Surname	Item
Luigi	Santocanale	33
Alan	Turing	21

$\bowtie$

Item	Description
33	Book
33	Livre
21	Machine

$=$

Name	Surname	Item	Description
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Operations on tables: the inner union

Name	Surname	Item
Luigi	Santocanale	33
Alan	Turing	21

$\cup$

Name	Surname	Sport
Diego	Maradona	Football
Usain	Bolt	Athletics

$=$

Name	Surname
Luigi	Santocanale
Alan	Turing
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Usain	Bolt

Lattices from databases

Proposition. [Spight & Tropashko, 2006] The set of tables, whose columns are indexed by a subset of A and values are from a set D, is a lattice, with natural join as meet and inner union as join.

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The relational lattices $\mathsf{R}(D, A)$

A a set of attributes, D a set of values.

An element of $\mathsf{R}(D, A)$:

- a pair (X, T) with $X \subseteq A$ and $T \subseteq D^X$.

We have

$$(X_1, T_1) \leq (X_2, T_2) \text{ iff } X_2 \subseteq X_1 \text{ and } T_1{\upharpoonright}_{X_2} \subseteq T_2 \,.$$

Meet and join

$$
\begin{aligned}
(X_1, T_1) \wedge (X_2, T_2) &:= (X_1 \cup X_2, T) \\
\text{where } T &= \{ f \mid f_{\restriction X_i} \in T_i, i = 1, 2 \} \\
&= i_{X_1 \cup X_2}(T_1) \cap i_{X_1 \cup X_2}(T_2) ,
\end{aligned}
$$

$$
\begin{aligned}
(X_1, T_1) \vee (X_2, T_2) &:= (X_1 \cap X_2, T) \\
\text{where } T &= \{ f \mid \exists i \in \{ 1, 2 \}, \exists g \in T_i \text{ s.t. } g_{\restriction X_1 \cap X_2} = f \} \\
&= T_1 \Vert_{X_1 \cap X_2} \cup T_2 \Vert_{X_1 \cap X_2} \ .
\end{aligned}
$$

Representation via closure operators

The Hamming/Priess_Crampe-Ribenboim ultrametric distance on D^A:

$$\delta(f, g) := \{ x \in A \mid f(x) \neq g(x) \} .$$

NB: this distance takes values in the join-semilattice $(P(A), \emptyset, \cup)$.

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A subset X of $A + D^A$ is *closed* if $\delta(f,g) \cup \{ g \} \subseteq X$ implies $f \in X$.

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Proposition. [Litak, Mikulás and Hidders 2015] $\mathrm{R}(D, A)$ is isomorphic to the lattice of closed subsets of $A + D^A$.

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Undecidable quasiequational theories

Theorem. [Litak, Mikulás and Hidders, 2015] The set of quasiequations in the signature $(\wedge, \vee, H)$ that are valid on relational lattices is undecidable.

We refine here this to:

Theorem. The set of quasiequations in the signature $(\wedge, \vee)$ that are valid on relational lattices is undecidable.

We actually prove a stronger result:

Theorem. It is undecidable whether a finite subdirectly irreducible lattice embeds into some $\mathrm{R}(D, A)$.

Related undecidable problems

Theorem. [Maddux 1980] The equational theory of 3-dimensional diagonal free cylindric algebras is undecidable.

Theorem. [Hirsch and Hodkinson 2001] It is not decidable whether a finite simple relation algebras embeds into a concrete one (a powerset of a binary product).

Theorem. [Hirsch, Hodkinson and Kurucz 2002] It is not decidable whether a finite mutimodal Kripke frame has a surjective p-morphism from a universal product frame.

Frames, universal product frames

A (multimodal Kripke) *A-frame* is a pair $(X, \{ R_a \mid a \in A \})$ with

- X a set, and
- $R_a \subseteq X \times X$, for each $a \in A$.

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A *universal* **S5***-product frame* is an A-frame $(X, \{ R_a \mid a \in A \})$ with

- $X = \prod_{a \in A} Y_a$,
- $\vec{x} R_a \vec{y}$ iff $\vec{x}_b = \vec{y}_b$, for each $b \neq a$.

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A *p-morphism* from $(X, \{ R_a \mid a \in A \})$ to $(X', \{ R'_a \mid a \in A \})$ is a function $f : X \to X'$ such that

- $x R_a y$ implies $f(x) R'_a f(y)$, for each $a \in A$,
- $f(x) R'_a y'$ implies $x R_a y$ for some $y \in X$ such that $f(y) = y'$, for each $a \in A$.

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The lattice of a frame

Let $\mathcal{F} = (X, \{ R_a \mid a \in A \})$ be a finite A-frame.

If $\alpha \subseteq A$, then we say that $Y \subseteq X$ is α-closed if

$$x_0 R_{a_1} x_1 R_{a_2} x_2 \ldots R_{a_n} x_n \in Y \text{ and } \{ a_1, \ldots, a_n \} \subseteq \alpha$$
$$\text{implies } x_0 \in Y\,.$$

We say that $Z \subseteq A + X$ is closed if $Z \cap X$ is $Z \cap A$-closed.

Definition. The lattice $\mathsf{L}(\mathcal{F})$ is the lattice of closed subsets of $A + X$.

The lattice of a frame

Let $\mathcal{F} = (X, \{ R_a \mid a \in A \})$ be a finite A-frame.

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Definition. The lattice $\mathsf{L}(\mathcal{F})$ is the lattice of closed subsets of $A + X$.

Theorem. A full rooted **S4** mutimodal frame $\mathcal{F}$ has a surjective p-morphism from a universal product frame iff $\mathsf{L}(\mathcal{F})$ embeds into a relational lattice.

The easy part: embeddings from p-morphisms

L extends to a contravariant functor.

Moreover if $X = \prod_{a \in A} D$ $(= D^A)$ and A is finite then $\mathsf{L}(\mathcal{F}) = \mathsf{R}(D, A)$.

The easy part: embeddings from p-morphisms

L extends to a contravariant functor.

Moreover if $X = \prod_{a \in A} D$ $(= D^A)$ and A is finite then $\mathsf{L}(\mathcal{F}) = \mathsf{R}(D, A)$.

Corollary. If a finite multimodal frame $\mathcal{F}$ has a surjective p-morphism from a universal product frame, then $\mathsf{L}(\mathcal{F})$ embeds into some $\mathsf{R}(D, A)$.

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Lattice embeddings into the $\mathsf{R}(D, B)$s

We study lattice embeddings of the form

$$i : \mathsf{L}(\mathcal{F}) \longrightarrow \mathsf{R}(D, B)$$

where $\mathcal{F}$ is an A-frame.

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where $\mathcal{F}$ is an A-frame.

We can suppose that:

1. $A = B$ is both the set of join-prime elements of $\mathsf{L}(\mathcal{F})$ and the set of join-prime elements of $\mathsf{R}(D, B)$ $(= \mathsf{R}(D, A))$;
2. i preserves $\bot, \top$, so $\mu \dashv i$ (use $\mathsf{L}(\mathcal{F})$ subdirectly irreducible).

$$
\begin{tikzcd}
X_\nu \arrow[d] \arrow[dr, dotted, "\mu"] & \\
D^A \arrow[r, dotted, "?"] \arrow[d] & X_{\mathcal{F}} \arrow[d] \\
A + D^A = J(\mathsf{R}(D, A)) \arrow[d] & A + X_{\mathcal{F}} = J(\mathsf{L}(\mathcal{F})) \arrow[d] \\
\mathsf{R}(D, A) \arrow[r, "\mu"] & \mathsf{L}(\mathcal{F}) \arrow[l, bend left, "i"]
\end{tikzcd}
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\mathsf{R}(D,A) \arrow[rr, "\mu"] & & \mathsf{L}(\mathcal{F}) \arrow[ll, bend left, "i"]
\end{tikzcd}
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For $f \in D^A$, we have $\mu(f) \in X_{\mathcal{F}}$ if and only if $f \in X_\nu$, where

$$X_\nu := \{ f \in D^A \mid \nu(f) = \emptyset \}, \quad \nu(f) := \{ j \in A \mid j \leq \mu(f) \}.$$

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For $f \in D^A$, we have $\mu(f) \in X_{\mathcal{F}}$ if and only if $f \in X_\nu$, where

$$X_\nu := \{\, f \in D^A \mid \nu(f) = \emptyset \,\}, \quad \nu(f) := \{\, j \in A \mid j \leq \mu(f) \,\}.$$

Moreover ν is a module on the space (D^A, δ).

Some theory of generalized ultrametric spaces

An ultrametric space over $P(A)$ is a pair (X, δ) such that

- $\delta(x, y) = \emptyset$ iff $x = y$,
- $\delta(x, z) \subseteq \delta(x, y) \cup \delta(y, z)$,
- $\delta(x, y) = \delta(y, x)$.

A space (X, δ) is *pairwise-complete* if

- $\delta(x, z) \subseteq \alpha \cup \beta$ implies $\delta(x, y) \subseteq \alpha$ and $\delta(y, z) \subseteq \beta$, for some $y \in X$,

A space (X, δ) is *spherically-complete* if every chain of balls has non empty intersection.

Universal product frames as GUMSs

Theorem. [Ackerman 2013] For a GUMS (X, δ) over $P(A)$, TFAE:

- (X, δ) is an injective object in the category of GUMS over $P(A)$,
- (X, δ) is pairwise-complete and spherically-complete,

Universal product frames as GUMSs

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Theorem. [LS] For a GUMS (X, δ) over $P(A)$, TFAE:

- (X, δ) is pairwise-complete and spherically-complete,
- (X, δ) are spaces of sections (universal product frames, Hamming graphs, dependent product types, ...)

Modules

An ultrametric space (X, δ) is a category enriched over $(P(A), \emptyset, \cup)$.

A *module* on (X, δ) is an enriched functor $v : (X, \delta) \to (P(A), \triangle)$. That is, a function $v : X \to P(A)$ such that:

$$v(x) \subseteq \delta(x, y) \cup v(y)\,.$$

Lemma

If (X, δ) is spherically-complete and pairwise-complete and $v : (X, \delta) \to P(A)$ is a module, then its kernel

$$X_v = \{\, x \in X \mid v(x) = \emptyset \,\}$$

induces a spherically-complete and pairwise-complete subspace of (X, δ).

Completing the proof of the converse

The subspace induced by

$$X_\nu = \{\, f \in D^A \mid \mu(f) \in X_{\mathcal{F}} \,\} = \{\, f \in D^A \mid \nu(f) = \emptyset \,\}\,,$$

is the kernel of a module, therefore it is pairwise-complete (and spherically-complete), that is, a universal product frame.

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Then

$$\mu_{\restriction X_\nu} : X_\nu \longrightarrow X_{\mathcal{F}}$$

yields the desired surjective map.

Completing the proof of the converse

The subspace induced by

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Then

$$\mu_{\restriction X_\nu} : X_\nu \longrightarrow X_{\mathcal{F}}$$

yields the desired surjective map.

This map is a p-morphism since (roughly) this property corresponds to μ preserving joins.

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Back to equations

The reduction (with results by KHH) also yields the following:

Proposition. If $\mathbf{card}A \geq 3$, then there exists a quasiequation that holds in all the finite $\mathrm{R}(D', A)$, but fails in $\mathrm{R}(D, A)$ when D is infinite.

Back to equations

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Theorem. If A or D is finite, then $\mathrm{R}(D, A)$ belongs to the variety generated by the finite $\mathrm{R}(D', A')$.

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Theorem. If A or D is finite, then $\mathrm{R}(D, A)$ belongs to the variety generated by the finite $\mathrm{R}(D', A')$.

- If A is finite, then $\mathrm{R}(D, A)$ is an algebraic lattice.

Back to equations

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Theorem. If A or D is finite, then $\mathrm{R}(D, A)$ belongs to the variety generated by the finite $\mathrm{R}(D', A')$.

- If A is finite, then $\mathrm{R}(D, A)$ is an algebraic lattice.
- If A is infinite, then $\mathrm{R}(D, A)$ is not an algebraic lattice.

Functorial properties

Let $f : A \to B$ be a (set theoretic function).
Then

$$\mathrm{R}(D, f) : \mathrm{R}(D, A) \to \mathrm{R}(D, B)$$

defined by

$$\mathrm{R}(D, f)(\alpha, X) := (\forall_f(\alpha), f^{*-1}(X))$$

makes $\mathrm{R}(D, -)$ into a functor from **Set** to $_\wedge$**SL**.

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Proposition. If D is finite, then the canonical map

$$\mathrm{R}(D, A) \to \lim_{Q \text{ a finite partition of Q}} \mathrm{R}(D, A/Q)$$

is injective and preserves finite joins.

From meet-semilattices to lattices

- The projective limit $\lim_Q \mathrm{R}(D, A/Q)$ is an algebraic lattice.
- Compact elements are of the form $j_Q(\beta, Y)$, for some $(\beta, Y) \in \mathrm{R}(D, A/Q)$, for some finite partition Q of A. Here j_Q is left adjoint to $\mathrm{R}(D, \pi_Q) : \mathrm{R}(D, A) \to \mathrm{R}(D, A/Q)$ with $\pi_Q : A \to A/Q$.

Proposition. If $\pi : A \to B$ is surjective, then the left adjoint to $\mathrm{R}(D, \pi)$ is a right adjoint (that is, it preserves meets).
Theorem. The projective limit $\lim_Q \mathrm{R}(D, A/Q)$ is (up to isomorphism) the ideal completion of inductive $\mathrm{colim}_Q \mathrm{R}(D, A/Q)$, where the latter lives in the category of lattices.

Thanks ! Questions ?

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