The annihilator of fuzzy subgroups

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Purpose

- To give an appropriate notion of annihilator of a fuzzy subgroup which enlarges the one given for crisp subgroups in Pontryagin duality theory.

Outline

- We start with the concept of a duality (X, X^*) for an abelian group X.
- Using the $\alpha-$cut representation of a fuzzy subgroup, we construct both the annihilator of a fuzzy subgroup of X and the inverse annihilator.
- We show some properties of the annihilator.
- We conclude with two examples of annihilators of fuzzy subgroups.

The notion of duality in abelian groups

Let X, X' two abelian groups. We say that they are in duality if there is a function

$$\langle \cdot, \cdot \rangle : X \times X' \to \mathbb{T}$$

- It is homomorphism on each component
- If $x \neq e_X$, there exists $x' \in X'$ such that $\langle x, x' \rangle \neq 1$
- If $x' \neq e_{X'}$, there exists $x \in X$ such that $\langle x, x' \rangle \neq 1$

The notion of annihilator in the crisp setting

The notion of annihilator is defined in the context of the following duality.

- Let X be an abelian group, by a character of X we mean a homomorphism from X into the unit circle of complex plane $\mathbb{T}$.
- The set of characters with pointwise multiplication is the group $\text{Hom}(X, \mathbb{T})$.
- A subgroup X^* of $\text{Hom}(X, \mathbb{T})$ separates points of X if for all $x \neq e_X$, there exists $\chi \in X^*$ such that $\chi(x) \neq 1$.

- If X is an abelian group and X^* is a subgroup of Hom$(X, \mathbb{T})$ which separates points,

$$\langle \cdot, \cdot \rangle : X \times X^* \to \mathbb{T}, \ [\langle x, \chi \rangle = \chi(x)]$$

The pair (X, X^*) is a duality

- Pontryagin duality assigns to a locally compact abelian group another locally compact abelian group which is the group $X^\wedge :=$ CHom$(X, \mathbb{T})$ endowed with the compact open topology and it is called the Pontryagin dual group of X.
- If X is discrete its Pontryagin dual is compact.

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- An abelian group X can be treated as a discrete group. In this case $X^* := \text{Hom}(X, \mathbb{T})$ coincides with $X^\wedge := \text{CHom}(X, \mathbb{T})$. Therefore $(X, \text{Hom}(X, \mathbb{T}))$ is the Pontryagin duality for the discrete group X.
- For example, the Pontryagin dual of the group of integer numbers $\mathbb{Z}$ is isomorphic to $\mathbb{T}$ by means of the mapping

$$\mathbb{T} \to \mathbb{Z}^\wedge$$

$$t \to (n \to t^n)$$

Question. Is there a way to go back and forth between subgroups of X and subgroups of $X^\wedge$ without loosing information?.

The answer is through the annihilator operator.

- Let G be a subgroup of X, the annihilator of G is the subgroup

$$G^\perp := \{\varphi \in X^* \mid \varphi(G) = 1\}. \tag{1}$$

- If L is a subgroup of X^*, the inverse annihilator of L is defined by

$$^\perp L := \{x \in X \mid \varphi(x) = 1, \forall \varphi \in L\}. \tag{2}$$

Proposition 1

Let (X, X^*) be a duality.

1. If G_1, G_2 are subgroups of X and $G_1 \subset G_2$, then $G_2^\perp \subset G_1^\perp$
2. If $\mathcal{J}$ is a totally ordered set and $\{G_j\}_{j\in\mathcal{J}}$ is an increasing family of subgroups of an abelian group X, $(\cup_{j\in\mathcal{J}} G_j)^\perp = \cap_{j\in\mathcal{J}} G_j^\perp$.
3. If $\mathcal{J}$ is arbitrary and $\{G_j\}_{j\in\mathcal{J}}$ is a decreasing family of subgroups of an abelian group X, $(\cap_{j\in\mathcal{J}} G_j)^\perp \geqslant \cup_{j\in\mathcal{J}} G_j^\perp$.

Proposition 2

1. The previous three properties also hold for the corresponding inverse annihilators statements.
2. Given the duality $(X, \mathsf{Hom}(X, \mathbb{T}))$, we have that

$$^{\perp}(G^{\perp}) = G$$

for all subgroups G of X.

Fuzzy subgroups.

X is a nonempty set, $\mathbb{I}$ is the unit interval $[0, 1]$ and $\mathbb{I}^X$ denote the family of all fuzzy sets on X.

Let G be a fuzzy subset of an abelian group X, we will say that G is a fuzzy subgroup of X if it satisfies

(FG1) $G(xy) \geq \min\{G(x), G(y)\} \; \forall x, y \in X$

(FG2) $G(x^{-1}) \geq G(x)$ for all $\forall x \in X$

A fuzzy subgroup G is determined by its (open) closed α-cuts: A fuzzy set A in a group X is a fuzzy subgroup of X if and only if each non-empty (open) closed α-cut of A is a subgroup of X.

- For each $A \in [0,1]^X$ and $\alpha \in [0,1]$,

  the closed α-cut of A is the set $A_\alpha = \{x \in X \mid A(x) \geq \alpha\}$ and

  the open α-cut is the set $A^\alpha = \{x \in X \mid A(x) > \alpha\}$.

- A fuzzy set can be recovered from its α-cuts:

  $$A(x) = \sup_{\alpha \in (0,1]} \{\alpha \cdot \chi_{A_\alpha}(x))\}$$

- A family $\{A_\alpha\}_{\alpha \in [0,1]}$ of subsets of X is the family of α-cuts of a fuzzy subset of X if and only if it satisfies:

  (i) $A_0 = X$

  (ii) $\alpha < \beta$ implies $A_\alpha \supseteq A_\beta$ $\forall \alpha, \beta \in [0,1]$

  (iii) $\bigcap_{\alpha < \beta} A_\alpha = A_\beta$ $\forall \beta \in (0,1]$

The annihilator of a fuzzy subgroup.

Let (X, X^*) be a duality and G a fuzzy subgroup of X.

We construct the annihilator of G in the following way:

$$(G^{\perp})_{\alpha} = \begin{cases} X^* & \text{if } \alpha = 0 \\ (G^{G(e)-\alpha})^{\perp} & \text{if } 0 < \alpha \leq G(e) \\ \emptyset & \text{if } \alpha > G(e) \end{cases} \tag{3}$$

The family $A_\alpha = (G^{\perp})_\alpha$, $\alpha \in [0,1]$ satisfies:

(i) $A_0 = X^*$,

(ii) $\alpha < \beta$ implies $A_\alpha \supset A_\beta$ $\forall \alpha, \beta \in [0,1]$,

(iii) $\bigcap_{\alpha<\beta} A_\alpha = A_\beta$ $\forall \beta \in (0,1]$

Hence, it is the family of $\alpha - cuts$ of a fuzzy subgroup $G^{\perp}$ of X^*.

The inverse annihilator of a fuzzy subgroup.

Let us now define the inverse annihilator for a fuzzy subgroup H of X^*

$$({}^{\perp}H)_\alpha = \begin{cases} X & \text{if } \alpha = 0 \\ {}^{\perp}(H^{H(e)-\alpha}) & \text{if } 0 < \alpha \leq H(e) \\ \emptyset & \text{if } \alpha > H(e) \end{cases} \tag{4}$$

Property of annihilators.

Given the duality $(X, \mathrm{Hom}(X, \mathbb{T}))$, and a fuzzy subgroup G of X, we have that ${}^{\perp}(G^{\perp}) = G$.

Proof If $\alpha = 0$, $G_0 = X$ and $\left[{}^{\perp}(G^{\perp})\right]_0 = X$ by definition.

If $\alpha > G(e)$ then $\alpha > G^{\perp}(e)$ and $\left[{}^{\perp}(G^{\perp})\right]_{\alpha} = \emptyset = G_{\alpha}$.

If $0 < \alpha \leq G(e)$:

$$\left[{}^{\perp}(G^{\perp})\right]_{\alpha} \overset{(1)}{=} {}^{\perp}\left[(G^{\perp})^{G^{\perp}(e)-\alpha}\right] \overset{(2)}{=} {}^{\perp}\left[\bigcup_{\beta > G^{\perp}(e)-\alpha} (G^{\perp})_{\beta}\right]$$

$$\overset{(3)}{=} \bigcap_{\beta > G^{\perp}(e)-\alpha} {}^{\perp}\left[(G^{\perp})_{\beta}\right] \overset{(4)}{=} \bigcap_{\beta > G^{\perp}(e)-\alpha} {}^{\perp}\left[(G^{G(e)-\beta})^{\perp}\right]$$

$$\overset{(5)}{=} \bigcap_{\beta > G(e)-\alpha} G^{G(e)-\beta} = \bigcap_{\alpha > G(e)-\beta} G^{G(e)-\beta} \overset{(6)}{=} G_{\alpha}$$

Behavior of the annihilator with respect to operations with fuzzy subgroups

1. If G_1, G_2 are fuzzy subgroups of an abelian group X satisfying $G_1 \subset G_2$ and $G_1(e) = G_2(e)$, then $G_2^{\perp} \subset G_1^{\perp}$.
2. If $\{G_j\}_{j\in\mathcal{J}}$ is a non increasing family of fuzzy subgroups and $G_i(e) = G_j(e)$ for all $i, j \in \mathcal{J}$, then $(\wedge_{j\in\mathcal{J}} G_J)^{\perp} \supset \vee_{j\in\mathcal{J}}(G_j)^{\perp}$
3. Let $\mathcal{J}$ be a totally ordered set, and $\{G_j\}_{j\in\mathcal{J}}$ an increasing family of fuzzy subgroups such that $G_i(e) = G_j(e)$ for all $i, j \in \mathcal{J}$, then $(\vee_{j\in\mathcal{J}} G_j)^{\perp} = \wedge_{j\in\mathcal{J}}(G_j)^{\perp}$

Example 1

Denote $C_{p^n} := \mathbb{Z}/p^n\mathbb{Z}$ the cyclic group of order p^n. Then

$$\{0\} = C_1 \subset C_p \subset C_{p^2} \subset \cdots \subset C_{p^{n-1}} \subset C_{p^n}$$

Given any sequence $1 \geq t_n \geq t_{n-1} \geq \cdots \geq t_1 \geq t_0 \geq 0$ we can define a fuzzy subgroup G of C_{p^n} as follows:

$$G(x) = \begin{cases} t_n & \text{if } x = 0 \\ t_{n-1} & \text{if } x \in C_p - \{0\} \\ t_{n-k} & \text{if } x \in C_{p^k} - C_{p^{k-1}} \\ t_0 & \text{if } x \in C_{p^n} - C_{p^{n-1}} \end{cases}$$

We are going to compute the annihilator of G in the duality $(C_{p^n}, \mathrm{Hom}(C_{p^n}, \mathbb{T}))$. $\mathrm{Hom}(C_{p^n}, \mathbb{T}) \cong C_{p^n}$

Every homomorphism $\varphi_m : C_{p^n} \to \mathbb{T}$ is defined by their image at the generator 1 of C_{p^n} by $\varphi_m(1) = e^{2\pi i m/p^n}$, $0 \le m \le p^n - 1$.

$$G^{\perp}(x) = \begin{cases} t_n - t_0 & \text{if } x = 0 \\ t_n - t_1 & \text{if } x \in C_p - \{0\} \\ t_n - t_k & \text{if } x \in C_{p^k} - C_{p^{k-1}} \\ t_n - t_{n-1} & \text{if } x \in C_{p^n} - C_{p^{n-1}} \end{cases}$$

Example 2

Consider the group of integer numbers $(\mathbb{Z}, +)$ and the duality $(\mathbb{Z}, \mathbb{T})$.

$$G(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Z} \setminus 2\mathbb{Z} \\ 1 - 1/n & \text{if } x = m2^n \text{ with } m \in \mathbb{Z} \setminus 2\mathbb{Z},\ n > 0 \\ 1 & \text{if } x = 0 \end{cases}$$

$$G^{\perp}(t) = \begin{cases} 0 & \text{if } t \notin \bigcup_{n \geq 1} C_{2^{n+1}} \\ \dfrac{1}{n} & \text{if } t \in C_{2^{n+1}} \setminus C_{2^n} \\ 1 & \text{if } t \in C_2 \end{cases}$$

where $(n\mathbb{Z})^{\perp} = C_n$ and C_n denotes the subgroup of $\mathbb{T}$ of the n-th roots of the unity.