Tensor products of Cuntz semigroups

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26. June 2017
TACL, Prague

The category **Cu** of abstract Cuntz semigroup

Recall:

- **Cu**-semigroup is domain with monoid structure such that addition is jointly Scott continuous and $\ll$-preserving:
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- **Cu**-morphism $f \colon S \to T$ is additive, $\ll$-preserving Scott continuous map:
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- Reflection functors transfer monoidal structure.

Category W of pre-completed Cuntz semigroups

- The predecessor set: $a^{\prec} := \{x \mid x \prec a\}$.

Definition

W-semigroup *is monoid with transitive relation $\prec$ such that:*

- $\prec$ *has interpolation:* $a^{\prec}$ *is upward directed.*

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Theorem

- **Cu** *is a full, reflective subcategory of* **W**.
- *Have completion functor* $\gamma\colon \mathbf{W} \to \mathbf{Cu}$.

Bimorphisms

$\otimes_{\mathrm{Cu}}$ should linearize bilinear maps:

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Lemma

- $S \otimes_{\mathrm{W}} T := (S \otimes_{\mathrm{alg}} T, \prec)$ *is* **W***-semigroup.*
- $S \times T \to S \otimes_{\mathrm{W}} T$ *is* **W***-bimorphism with universal property:*

$$\mathrm{W}(S \otimes_{\mathrm{W}} T, R) \xrightarrow{\cong} \mathrm{BiW}(S \times T, R).$$

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- **Cu** *is a symmetric, monoidal category.*

Proof.

$$\begin{aligned} \mathrm{BiCu}(S \times T, R) &\cong \mathrm{BiW}(S \times T, R) \cong \mathrm{W}(S \otimes_{\mathrm{W}} T, R) \\ &\cong \mathrm{Cu}(\gamma(S \otimes_{\mathrm{W}} T), R) = \mathrm{Cu}(S \otimes_{\mathrm{Cu}} T, R). \quad \square \end{aligned}$$

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- $\{0, \infty\} \otimes S \cong \mathrm{Lat}(S)$ - lattice of ideals in S (Scott-closed submonoids).

Cu is closed

- Category **Q** such that **Cu** $\subseteq$ **Q** full, hereditary.
- Functor $\tau\colon$ **Q** $\to$ **Cu** that is coreflection:

$$\mathrm{Q}(T, P) \cong \mathrm{Cu}(T, \tau(P)).$$

- **Q** admits right adjoint for its bimorphism functor:

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- *$[\![T,\_]\!]$ is right adjoint to $\_\otimes_{\mathrm{Cu}} T$:*

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- ***Cu** is a closed, symmetric, monoidal category.*

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- For $Z = \mathbb{N} \cup (0, \infty]$, have $[\![Z, Z]\!] = Z$.
- For $M := \mathrm{Cu}(\mathrm{II}_1\text{-factor})$, have $[\![\overline{\mathbb{R}}_+, \overline{\mathbb{R}}_+]\!] = M$, $[\![M, M]\!] = M$.

Thank you.