

Weak Fraïssé categories

Wiesław Kubiś

Institute of Mathematics, Czech Academy of Sciences
and
Cardinal Stefan Wyszyński University in Warsaw, Poland

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Sources, motivation

- ① A. Krawczyk, W. Kubiś, *Games on finitely generated structures*, preprint, (arXiv:1701.05756)
- ② A. Krawczyk, A. Kruckman, W. Kubiś, A. Panagiotopoulos, *Examples of weak amalgamation classes*, preprint
- ③ A. Kruckman, *Infinitary Limits of Finite Structures*, PhD thesis, University of California, Berkeley 2016
- ④ W. Kubiś, *Banach-Mazur game played in partially ordered sets*, Banach Center Publications 108 (2016) 151–160 (arXiv:1505.01094)
- ⑤ A. Ivanov, *Generic expansions of ω -categorical structures and semantics of generalized quantifiers*, J. Symbolic Logic 64 (1999) 775–789
- ⑥ R. Fraïssé, *Sur l'extension aux relations de quelques propriétés des ordres*, Ann. Sci. Ecole Norm. Sup. (3) 71 (1954) 363–388

Categories

A *category* \mathcal{K} consists of

- a class of objects $\text{Obj}(\mathcal{K})$,
- a class of arrows $\bigcup_{A,B \in \text{Obj}(\mathcal{K})} \mathcal{K}(A, B)$, where $f \in \mathcal{K}(A, B)$ means A is the *domain* of f and B is the *codomain* of f ,
- a partial associative composition operation \circ defined on arrows, where $f \circ g$ is defined \iff the domain of g coincides with the domain of f .

Furthermore, for each $A \in \text{Obj}(\mathcal{K})$ there is an *identity* $\text{id}_A \in \mathcal{K}(A, A)$ satisfying $\text{id}_A \circ g = g$ and $f \circ \text{id}_A = f$ for $f \in \mathcal{K}(A, X)$, $g \in \mathcal{K}(Y, A)$, $X, Y \in \text{Obj}(\mathcal{K})$.

The setup

\mathcal{K} is a fixed category, $\mathcal{L} \supseteq \mathcal{K}$ is a bigger category such that \mathcal{K} is full in \mathcal{L} and the following conditions are satisfied:

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- (L2) Every sequence in \mathcal{K} has a co-limit in \mathcal{L} .
- (L3) Every \mathcal{K} -object is ω -small in \mathcal{L} .

Sequences

Definition

A **sequence** is a covariant functor $\vec{X}: \omega \rightarrow \mathcal{K}$.

$$x_0 \xrightarrow{x_0^1} x_1 \xrightarrow{x_1^2} x_2 \longrightarrow \dots$$

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More generally, after Odd's move finishing with an object A_{2k-1} , Eve chooses $A_{2k} \in \text{Obj}(\mathfrak{K})$ together with a \mathfrak{K} -arrow $a_{2k-1}^{2k}: A_{2k-1} \rightarrow A_{2k}$.

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Next, Odd chooses $A_{2k+1} \in \text{Obj}(\mathfrak{K})$ together with a \mathfrak{K} -arrow $a_{2k}^{2k+1}: A_{2k} \rightarrow A_{2k+1}$. And so on...

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The result of a play is a sequence \vec{a} :

$$A_0 \xrightarrow{a_0^1} A_1 \longrightarrow \cdots \longrightarrow A_{2k-1} \xrightarrow{a_{2k-1}^{2k}} A_{2k} \longrightarrow \cdots$$

Generic objects

Definition

We say that $U \in \text{Obj}(\mathfrak{L})$ is \mathfrak{K} -generic if Odd has a strategy in the Banach-Mazur game such that the colimit of the resulting sequence \vec{a} is always isomorphic to U , no matter how Eve plays.

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Proposition

A \mathfrak{K} -generic object, if exists, is unique up to isomorphism.

Variants of amalgamations

Definition

We say that \mathfrak{K} has **amalgamations at** $Z \in \text{Obj}(\mathfrak{K})$ if for every \mathfrak{K} -arrows $f: Z \rightarrow X$, $g: Z \rightarrow Y$ there exist \mathfrak{K} -arrows $f': X \rightarrow W$, $g': Y \rightarrow W$ such that $f' \circ f = g' \circ g$.

$$\begin{array}{ccc} Y & \xrightarrow{g} & W \\ g \uparrow & & \uparrow f' \\ Z & \xrightarrow{f} & X \end{array}$$

We say that \mathfrak{K} has the **amalgamation property (AP)** if it has the amalgamation property at every $Z \in \text{Obj}(\mathfrak{K})$.

Definition

We say that \mathfrak{K} has the **cofinal amalgamation property (CAP)** if for every $Z \in \text{Obj}(\mathfrak{K})$ there is a \mathfrak{K} -arrow $e: Z \rightarrow Z'$ such that \mathfrak{K} has amalgamations at Z' .

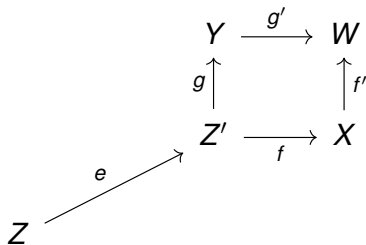
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Definition (Ivanov, 1999)

We say that \mathfrak{K} has the **weak amalgamation property (WAP)** if for every $Z \in \text{Obj}(\mathfrak{K})$ there is a \mathfrak{K} -arrow $e: Z \rightarrow Z'$ such that for every \mathfrak{K} -arrows $f: Z' \rightarrow X$, $g: Z' \rightarrow Y$ there exist \mathfrak{K} -arrows $f': X \rightarrow W$, $g': Y \rightarrow W$ such that $f' \circ f \circ e = g' \circ g \circ e$.

CAP and WAP



Directedness and weak domination

Definition

We say that \mathcal{K} is **directed** if for every $X, Y \in \text{Obj}(\mathcal{K})$ there exist \mathcal{K} -arrows f, g such that $\text{dom}(f) = X$, $\text{dom}(g) = Y$, and $\text{cod}(f) = \text{cod}(g)$.

Directedness and weak domination

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Definition

We say that $\mathcal{F} \subseteq \mathfrak{K}$ is **weakly dominating** if

- 1 For every $X \in \text{Obj}(\mathfrak{K})$ there is $f \in \mathcal{F}$ such that $\text{dom}(f) = X$ and $\text{cod}(f) \in \text{Obj}(\mathfrak{G})$.

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- 1 For every $X \in \text{Obj}(\mathfrak{K})$ there is $f \in \mathfrak{K}$ such that $\text{dom}(f) = X$ and $\text{cod}(f) \in \text{Obj}(\mathcal{F})$.
- 2 For every $Y \in \text{Obj}(\mathcal{F})$ there exists an \mathcal{F} -arrow $j: Y \rightarrow Y'$ such that for every \mathfrak{K} -arrow $f: Y' \rightarrow Z$ there is a \mathfrak{K} -arrow $g: Z \rightarrow W$ satisfying $g \circ f \circ j \in \mathcal{F}$.

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- is dominated by a countable subcategory.

Theorem

Assume \mathcal{K} is a weak Fraïssé category. Then there exists a \mathcal{K} -generic object $U \in \text{Obj}(\mathcal{L})$.

The object U will be called the **limit** of \mathcal{K} .

Weak injectivity

Definition

We say that $V \in \text{Obj}(\mathcal{L})$ is **weakly \mathcal{K} -injective** if for every $A \in \text{Obj}(\mathcal{K})$, for every \mathcal{L} -arrow $e: A \rightarrow V$ there exists a \mathcal{K} -arrow $i: A \rightarrow B$ such that for every \mathcal{K} -arrow $f: B \rightarrow Y$ there is an \mathcal{L} -arrow $g: Y \rightarrow V$ satisfying $g \circ f \circ i = e$.

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \xrightarrow{f} & Y \\ e \downarrow & & & \nearrow g & \\ V & & & & \end{array}$$

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V is **\mathcal{K} -injective** if we can always take $i := \text{id}_A$.

Definition

We say that $V \in \text{Obj}(\mathfrak{L})$ is \mathfrak{K} -**cofinal** if for every $X \in \text{Obj}(\mathfrak{K})$ there exists an \mathfrak{L} -arrow $e: X \rightarrow V$.

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Theorem

Assume \mathfrak{K} is a weak Fraïssé category and $V \in \text{Obj}(\mathfrak{L})$. The following assertions are equivalent.

- 1 V is \mathfrak{K} -cofinal and weakly \mathfrak{K} -injective.
- 2 V is \mathfrak{K} -generic.

Homogeneity

Definition

We say that $U \in \text{Obj}(\mathfrak{L})$ is **\mathfrak{K} -homogeneous** if for every $A \in \text{Obj}(\mathfrak{K})$ for every \mathfrak{K} -arrows $i_0: A \rightarrow U$, $i_1: A \rightarrow U$ there exists an automorphism $h: U \rightarrow U$ such that $i_1 = h \circ i_0$.

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Definition

The **age** of $U \in \text{Obj}(\mathfrak{L})$ is

$$\text{Age}(U) := \{A \in \text{Obj}(\mathfrak{K}) : (\exists e \in \mathfrak{L}) \ e: A \rightarrow U\}.$$

Proposition

Assume U is \mathfrak{K} -homogeneous. Then $\text{Age}(U)$ has the amalgamation property.

Weak homogeneity

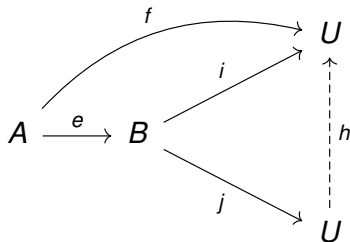
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We say that $U \in \text{Obj}(\mathfrak{L})$ is **weakly \mathfrak{K} -homogeneous** if for every $A \in \text{Obj}(\mathfrak{K})$, for every \mathfrak{L} -arrow $f: A \rightarrow U$ there exists a \mathfrak{K} -arrow $e: A \rightarrow B$ and an \mathfrak{L} -arrow $i: B \rightarrow U$ such that $f = i \circ e$ and for every \mathfrak{L} -arrow $j: B \rightarrow U$ there is an automorphism $h: U \rightarrow U$ satisfying $f = h \circ j \circ e$.

Weak homogeneity

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(The triangle may not be commutative!)

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Assume U is weakly \mathfrak{K} -homogeneous and $\text{Age}(U)$ has the amalgamation property. Then U is $\text{Age}(U)$ -homogeneous.

Main result

Denote by $\text{BM}(\mathfrak{K}, V)$ the Banach-Mazur game defined above, where Odd **wins** if and only if the resulting sequence is isomorphic to V .

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Theorem

Assume \mathfrak{K} is locally countable. Given an $V \in \text{Obj}(\mathfrak{L})$, the following properties are equivalent.

- ❶ *V is the limit of \mathfrak{K} (in particular, \mathfrak{K} is a weak Fraïssé category).*
- ❷ *Odd has a winning strategy in $\mathbf{BM}(\mathfrak{K}, V)$ (i.e. V is \mathfrak{K} -generic).*
- ❸ *Eve does not have a winning strategy in $\mathbf{BM}(\mathfrak{K}, V)$.*

The model-theoretic case is due to Krawczyk & K.

Applications

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- More abstract categories?

Three examples

Example 1

\mathfrak{K} = the category of all finite graphs with embeddings.

\mathfrak{L} = the category of all countable graphs with embeddings.

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The \mathfrak{K} -generic object is the **Rado graph** (found by Rado and independently by Erdős & Renyi).

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The \mathfrak{K} -generic object is the countable tree in which each vertex has infinite degree.

Example 3 (Krawczyk, Kruckman, Panagiotopoulos, K.)

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Claim

\mathfrak{K} has the WAP, not the CAP.

Further research: Generic objects in continuous category theory.

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Thank you for your attention!