Weak Fraïssé categories

Wiesław Kubiś

Institute of Mathematics, Czech Academy of Sciences
and
Cardinal Stefan Wyszyński University in Warsaw, Poland

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Sources, motivation

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3. A. Kruckman, *Infinitary Limits of Finite Structures*, PhD thesis, University of California, Berkeley 2016
4. W. Kubiś, *Banach-Mazur game played in partially ordered sets*, Banach Center Publications 108 (2016) 151–160 (arXiv:1505.01094)
5. A. Ivanov, *Generic expansions of ω-categorical structures and semantics of generalized quantifiers*, J. Symbolic Logic 64 (1999) 775–789
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Categories

A *category* $\mathfrak{K}$ consists of

- a class of objects $\mathrm{Obj}(\mathfrak{K})$,
- a class of arrows $\bigcup_{A,B\in\mathrm{Obj}(\mathfrak{K})}$, where $f \in \mathfrak{K}(A, B)$ means A is the *domain* of f and B is the *codomain* of f,
- a partial associative composition operation $\circ$ defined on arrows, where $f \circ g$ is defined $\iff$ the domain of g coincides with the domain of f.

Furthermore, for each $A \in \mathrm{Obj}(\mathfrak{K})$ there is an *identity* $\mathrm{id}_A \in \mathfrak{K}(A, A)$ satisfying $\mathrm{id}_A \circ g = g$ and $f \circ \mathrm{id}_A = f$ for $f \in \mathfrak{K}(A, X)$, $g \in \mathfrak{K}(Y, A)$, $X, Y \in \mathrm{Obj}(\mathfrak{K})$.

The setup

$\mathfrak{K}$ is a fixed category, $\mathfrak{L} \supseteq \mathfrak{K}$ is a bigger category such that $\mathfrak{K}$ is full in $\mathfrak{L}$ and the following conditions are satisfied:

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(L2) Every sequence in $\mathfrak{K}$ has a co-limit in $\mathfrak{L}$.

(L3) Every $\mathfrak{K}$-object is ω-small in $\mathfrak{L}$.

Sequences

Definition

A sequence is a covariant functor $\vec{x} \colon \omega \to \mathfrak{K}$.

$$x_0 \xrightarrow{x_0^1} x_1 \xrightarrow{x_1^2} x_2 \longrightarrow \cdots$$

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Then Odd chooses $A_1 \in \mathrm{Obj}(\mathfrak{K})$ together with a $\mathfrak{K}$-arrow $a_0^1 \colon A_0 \to A_1$.

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Then Odd chooses $A_1 \in \mathrm{Obj}(\mathfrak{K})$ together with a $\mathfrak{K}$-arrow $a_0^1 \colon A_0 \to A_1$.
More generally, after Odd's move finishing with an object A_{2k-1}, Eve chooses $A_{2k} \in \mathrm{Obj}(\mathfrak{K})$ together with a $\mathfrak{K}$-arrow $a_{2k-1}^{2k} \colon A_{2k-1} \to A_{2k}$.

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Next, Odd chooses $A_{2k+1} \in \mathrm{Obj}(\mathfrak{K})$ together with a $\mathfrak{K}$-arrow $a_{2k}^{2k+1} \colon A_{2k} \to A_{2k+1}$. And so on...

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The result of a play is a sequence $\vec{a}$:

$$A_0 \xrightarrow{a_0^1} A_1 \longrightarrow \cdots \longrightarrow A_{2k-1} \xrightarrow{a_{2k-1}^{2k}} A_{2k} \longrightarrow \cdots$$

Generic objects

Definition

We say that $U \in \mathrm{Obj}(\mathfrak{L})$ is $\mathfrak{K}$-generic if Odd has a strategy in the Banach-Mazur game such that the colimit of the resulting sequence $\vec{a}$ is always isomorphic to U, no matter how Eve plays.

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Proposition

A $\mathfrak{K}$-generic object, if exists, is unique up to isomorphism.

Variants of amalgamations

Definition

We say that $\mathfrak{K}$ has amalgamations at $Z \in \mathrm{Obj}(\mathfrak{K})$ if for every $\mathfrak{K}$-arrows $f\colon Z \to X$, $g\colon Z \to Y$ there exist $\mathfrak{K}$-arrows $f'\colon X \to W$, $g'\colon Y \to W$ such that $f' \circ f = g' \circ g$.

$$\begin{tikzcd} Y \arrow[r, "g"] & W \\ Z \arrow[u, "g"] \arrow[r, "f"'] & X \arrow[u, "f'"'] \end{tikzcd}$$

We say that $\mathfrak{K}$ has the amalgamation property (AP) if it has the amalgamation property at every $Z \in \mathrm{Obj}(\mathfrak{K})$.

Definition

We say that $\mathfrak{K}$ has the cofinal amalgamation property (CAP) if for every $Z \in \mathrm{Obj}(\mathfrak{K})$ there is a $\mathfrak{K}$-arrow $e \colon Z \to Z'$ such that $\mathfrak{K}$ has amalgamations at Z'.

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Definition (Ivanov, 1999)

We say that $\mathfrak{K}$ has the weak amalgamation property (WAP) if for every $Z \in \mathrm{Obj}(\mathfrak{K})$ there is a $\mathfrak{K}$-arrow $e \colon Z \to Z'$ such that for every $\mathfrak{K}$-arrows $f \colon Z' \to X$, $g \colon Z' \to Y$ there exist $\mathfrak{K}$-arrows $f' \colon X \to W$, $g' \colon Y \to W$ such that $f' \circ f \circ e = g' \circ g \circ e$.

CAP and WAP

$$\begin{tikzcd} & Y \arrow[r, "g'"] & W \\ & Z' \arrow[u, "g"] \arrow[r, "f"'] & X \arrow[u, "f'"'] \\ Z \arrow[ur, "e"] & & \end{tikzcd}$$

Directedness and weak domination

Definition

We say that $\mathfrak{K}$ is directed if for every $X, Y \in \mathrm{Obj}(\mathfrak{K})$ there exist $\mathfrak{K}$-arrows f, g such that $\mathrm{dom}(f) = X$, $\mathrm{dom}(g) = Y$, and $\mathrm{cod}(f) = \mathrm{cod}(g)$.

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Definition

We say that $\mathscr{F} \subseteq \mathfrak{K}$ is weakly dominating if

1. For every $X \in \mathrm{Obj}(\mathfrak{K})$ there is $f \in \mathfrak{K}$ such that $\mathrm{dom}(f) = X$ and $\mathrm{cod}(f) \in \mathrm{Obj}(\mathfrak{S})$.

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2. For every $Y \in \mathrm{Obj}(\mathfrak{S})$ there exists an $\mathfrak{S}$-arrow $j \colon Y \to Y'$ such that for every $\mathfrak{K}$-arrow $f \colon Y' \to Z$ there is a $\mathfrak{K}$-arrow $g \colon Z \to W$ satisfying $g \circ f \circ j \in \mathfrak{S}$.

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- is dominated by a countable subcategory.

Theorem

Assume $\mathfrak{K}$ is a weak Fraïssé category. Then there exists a $\mathfrak{K}$-generic object $U \in \mathrm{Obj}(\mathfrak{L})$.

The object U will be called the limit of $\mathfrak{K}$.

Weak injectivity

Definition

We say that $V \in \mathrm{Obj}(\mathfrak{L})$ is weakly $\mathfrak{K}$-injective if for every $A \in \mathrm{Obj}(\mathfrak{K})$, for every $\mathfrak{L}$-arrow $e \colon A \to V$ there exists a $\mathfrak{K}$-arrow $i \colon A \to B$ such that for every $\mathfrak{K}$-arrow $f \colon B \to Y$ there is an $\mathfrak{L}$-arrow $g \colon Y \to V$ satisfying $g \circ f \circ i = e$.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "e"'] & B \arrow[r, "f"] & Y \arrow[dll, dashed, "g"] \\ V & & \end{tikzcd}$$

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V is $\mathfrak{K}$-injective if we can always take $i := \mathrm{id}_A$.

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Theorem

Assume $\mathfrak{K}$ is a weak Fraïssé category and $V \in \mathrm{Obj}(\mathfrak{L})$. The following assertions are equivalent.

1. *V is $\mathfrak{K}$-cofinal and weakly $\mathfrak{K}$-injective.*
2. *V is $\mathfrak{K}$-generic.*

Homogeneity

Definition

We say that $U \in \mathrm{Obj}(\mathfrak{L})$ is $\mathfrak{K}$-homogeneous if for every $A \in \mathrm{Obj}(\mathfrak{K})$ for every $\mathfrak{K}$-arrows $i_0 \colon A \to U$, $i_1 \colon A \to U$ there exists an automorphism $h \colon U \to U$ such that $i_1 = h \circ i_0$.

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Definition

The age of $U \in \mathrm{Obj}(\mathfrak{L})$ is

$$\mathrm{Age}(U) := \{A \in \mathrm{Obj}(\mathfrak{K}) \colon (\exists\, e \in \mathfrak{L})\ \ e \colon A \to U\}.$$

Proposition

Assume U is $\mathfrak{K}$-homogeneous. Then $\mathrm{Age}(U)$ has the amalgamation property.

Weak homogeneity

Definition

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Weak homogeneity

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$$\begin{tikzcd}
 & & U \\
A \arrow[r, "e"] \arrow[urr, bend left, "f"] & B \arrow[ur, "i"] \arrow[dr, "j"'] & \\
 & & U \arrow[uu, dashed, "h"']
\end{tikzcd}$$

(The triangle may not be commutative!)

Proposition

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Assume U is weakly $\mathfrak{K}$-homogeneous and $\operatorname{Age}(U)$ has the amalgamation property. Then U is $\operatorname{Age}(U)$-homogeneous.

Main result

Denote by BM $(\mathfrak{K}, V)$ the Banach-Mazur game defined above, where Odd wins if and only if the resulting sequence is isomorphic to V.

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Theorem

Assume $\mathfrak{K}$ is locally countable. Given an $V \in \mathrm{Obj}(\mathfrak{L})$, the following properties are equivalent.

1. *V is the limit of $\mathfrak{K}$ (in particular, $\mathfrak{K}$ is a weak Fraïssé category).*
2. *Odd has a winning strategy in* BM $(\mathfrak{K}, V)$ *(i.e. V is $\mathfrak{K}$-generic).*
3. *Eve does not have a winning strategy in* BM $(\mathfrak{K}, V)$.

The model-theoretic case is due to Krawczyk & K.

Applications

- Classical Fraïssé theory.

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- Inverse Fraïssé theory (Irwin & Solecki).

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- More abstract categories?

Three examples

Example 1

$\mathfrak{K}$ = the category of all finite graphs with embeddings.
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Claim

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The $\mathfrak{K}$-generic object is the Rado graph (found by Rado and independently by Erdős & Renyi).

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The $\mathfrak{K}$-generic object is the countable tree in which each vertex has infinite degree.

Example 3 (Krawczyk, Kruckman, Panagiotopoulos, K.)

$\mathfrak{K}$ = all finite cycle-free graphs in which no two vertices of degree > 2 are adjacent (with embeddings).

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Claim

$\mathfrak{K}$ has the WAP, not the CAP.

Further research: Generic objects in continuous category theory.

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Thank you for your attention!