Lawson topology as the space of located subsets

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The space of located subsets

We are interested in spaces of the located subsets of some structures.

**Examples**

- Extended Dedekind reals \((L, U)\), i.e. extended with \(+\infty, -\infty\).
  - \(q \in U \iff (\exists q' < q) q' \in U\),
  - \(p \in L \iff (\exists p' > p) p' \in L\),
  - \(L \cap U = \emptyset\),
  - \(p < q \implies p \notin U \lor q \in U\) (locatedness).

An extended Dedekind reals \((L, U)\) is equivalent to a located (possibly unbounded) upper real \(U\).

- \(q \in U \iff (\exists q' < q) q' \in U\),
- \(p < q \implies p \notin U \lor q \in U\) (locatedness)

- Compact (including \(\emptyset\)) subsets of a compact metric space \((X, d)\) with the Hausdorff metric whose values are in the extended reals.

\[
d(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b) \right\}.
\]
1. Is there a geometric theory (over some given structure) whose models are the located subsets of that structure?

2. If so, what is the locale presented by the theory?

A **geometric theory** $T = (P, R)$ over a set $P$ of propositional symbols is a set $R$ of axioms of the form

$$p_0 \land \cdots \land p_{n-1} \vdash \bigvee_{i \in I} q^i_0 \land \cdots \land q^i_{n_i-1}.$$ 

A **model** (ideal) of $T$ is subset $\alpha \subseteq P$ such that

$$\{p_0, \ldots, p_{n-1}\} \subseteq \alpha \implies (\exists i \in I) \{q^i_0, \ldots, q^i_{n_i-1}\} \subseteq \alpha$$

for all axioms $p_0 \land \cdots \land p_{n-1} \vdash \bigvee_{i \in I} q^i_0 \land \cdots \land q^i_{n_i-1}$ in $T$. 


Outline

1. Located subsets of continuous lattices
2. Examples of located subsets
3. Spaces of located subsets
4. Lawson topologies
An **continuous cover** is a structure $S = (S, \triangleleft, \text{wb})$ where $\triangleleft \subseteq S \times \text{Pow}(S)$ is a **cover** satisfying

$$
\begin{align*}
    a \in U & \quad \Rightarrow \quad a \triangleleft U, \\
    a \triangleleft U & \quad \Rightarrow \quad a \triangleleft V, \\
    U \triangleleft V & \quad \overset{\text{def}}{\iff} \quad (\forall a \in U) a \triangleleft V,
\end{align*}
$$

and $\text{wb}$ is function $\text{wb} : S \rightarrow \text{Pow}(S)$ such that

1. $a \triangleleft \text{wb}(a)$,
2. $(\forall b \in \text{wb}(a)) b \ll a$.

Here, $\ll$ is the **way-below** relation:

$$
a \ll b \overset{\text{def}}{\iff} (\forall U \subseteq S) [b \triangleleft U \implies (\exists A \in \text{Fin}(U)) a \triangleleft A].
$$

We have $a \ll b \iff (\exists A \in \text{Fin}(S)) a \triangleleft A \land A \subseteq \text{wb}(b)$.

The $\text{Sat}(S) \overset{\text{def}}{=} \{ A U \mid U \subseteq S \}$ where $A U = \{ a \in S \mid a \triangleleft U \}$ forms a **continuous lattice** with a base $\{ A B \mid B \in \text{Fin}(S) \}$. 

A **perfect map** between continuous covers $S = (S, \sqsubseteq, \text{wb})$ and $S' = (S', \sqsubseteq', \text{wb}')$ is a relation $r \subseteq S \times S'$ such that

1. $a \sqsubseteq' U \implies r^\rightarrow \{a\} \sqsubseteq r^\rightarrow U$

2. $a \ll' b \implies r^\rightarrow \{a\} \ll r^\rightarrow \{b\}$

Let $\text{CCov}$ be the category of continuous covers and perfect maps.

**Remark**

A perfect map $S \to S'$ between continuous covers corresponds to a Scott continuous map $f : \text{Sat}(S) \to \text{Sat}(S')$ that has a left adjoint.
Fix a continuous cover $S = (S, \prec, \text{wb})$. A subset $V \subseteq S$ is **splitting** if

$$a \prec U \& a \in V \implies (\exists b \in U) b \in V.$$  

**Lemma.** A subset $V \subseteq S$ is splitting iff

1. $a \prec \{a_0, \ldots, a_{n-1}\} \& a \in V \implies (\exists i < n) a_i \in V,$
2. $a \in V \implies (\exists b \ll a) b \in V.$
Located subsets of continuous covers

Fix a continuous cover $S = (S, \prec, \text{wb})$. A subset $V \subseteq S$ is **splitting** if

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2. $a \in V \implies (\exists b \ll a) b \in V.$

A splitting subset $V \subseteq S$ is a **located** if $a \ll b \implies a \notin V \lor b \in V$.

**Lemma.** A subset $V \subseteq S$ is located iff $a \in \text{wb}(b) \implies a \notin V \lor b \in V$. 


Located subsets of continuous covers

Fix a continuous cover $S = (S, \triangleleft, \text{wb})$. A subset $V \subseteq S$ is splitting if

$$a \triangleleft U \& a \in V \implies (\exists b \in U) b \in V.$$ 

**Lemma.** A subset $V \subseteq S$ is splitting iff

1. $a \triangleleft \{a_0, \ldots, a_{n-1}\} \& a \in V \implies (\exists i < n) a_i \in V,$
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A splitting subset $V \subseteq S$ is a located if $a \ll b \implies a \notin V \lor b \in V.$

**Lemma.** A subset $V \subseteq S$ is located iff $a \in \text{wb}(b) \implies a \notin V \lor b \in V.$

**Proposition.** “The located subsets of $S$” $= \text{CCov}(1, S).$
Examples
Examples of located subsets

Example (Scott topology on $\text{Pow}(\mathbb{N})$)

$\mathcal{P}\omega = (\text{Fin}(\mathbb{N}), \triangleleft_\omega, \text{wb})$ where

\[
A \triangleleft_\omega U \iff (\exists B \in U) B \subseteq A,
\]

\[
\text{wb}(A) \overset{\text{def}}{=} \{ B \in \text{Fin}(S) \mid A \subseteq B \}.
\]

- $V \subseteq \text{Fin}(\mathbb{N})$ is splitting iff it is closed downwards w.r.t. $\subseteq$.
- A splitting subset $V$ is located iff it is detachable (NB. $A \ll A$).
Examples of located subsets

Example (Scott topology on the bounded upper reals)

\[ \mathcal{R}^u = (\mathbb{Q}, \triangleleft_u, \text{wb}) \]

where

\[ q \triangleleft_u U \overset{\text{def}}{\iff} (\forall p < q) \ (\exists q' \in U) \ p < q', \]

\[ \text{wb}(q) \overset{\text{def}}{=} \{ p \in \mathbb{Q} | p < q \}. \]

- \( V \subseteq \mathbb{Q} \) is splitting iff it is an upper real, i.e.

\[ q \in V \iff (\exists p < q) \ p \in V. \]

- A splitting subset \( V \) is located iff it is a located upper real (extended real), i.e.

\[ p < q \implies p \notin V \lor q \in V. \]
Examples of located subsets

Example (Binary tree $\mathcal{C}$ (Formal Cantor space))

$\mathcal{C} = (\{0, 1\}^\ast, \triangleleft_{\mathcal{C}}, \text{wb})$ where

\[
a \triangleleft_{\mathcal{C}} U \overset{\text{def}}{\iff} (\exists k \in \mathbb{N}) (\forall c \in a[k]) (\exists b \in U) b \triangleleft c
\]

$\iff U$ is a uniform bar of $a$.

\[
a[k] \overset{\text{def}}{=} \{ a \ast b \mid |b| = k \},
\]

\[
\text{wb}(a) \overset{\text{def}}{=} \{ b \in \{0, 1\}^\ast \mid a \triangleleft b \}.
\]

$\triangleright V \subseteq \{0, 1\}^\ast$ is splitting iff $a \in V \iff (\exists i \in \{0, 1\}) a \ast \langle i \rangle \in V$.

$\triangleright$ A splitting subset $V$ is located iff it is detachable (NB. $a \ll a$), i.e. it is a (possibly empty) “spread”.
Examples of located subsets

Example (Locally compact metric spaces, Palmgren (2007))

Given a (Bishop) locally compact metric space \((X, d)\), its **localic completion** is the continuous cover \(\mathcal{M}(X) = (M_X, \triangleleft_X, \text{wb})\) where

- \(M_X \overset{\text{def}}{=} X \times \mathbb{Q}^+ = \{\,(b(x, \varepsilon)) \mid x \in X \& \varepsilon \in \mathbb{Q}^+\} \) with an order \(b(x, \varepsilon)_X < b(y, \delta) \overset{\text{def}}{=} d(x, y) + \varepsilon < \delta\).

- \(a \triangleleft_X U \overset{\text{def}}{=} (\forall b <_X a) (\exists A \in \text{Fin}(U)) (\exists \theta \in \mathbb{Q}^+) b \sqsubseteq_\theta A, b \sqsubseteq_\theta A \overset{\text{def}}{=} (\forall b(x, \varepsilon) <_X b) \varepsilon < \theta \rightarrow (\exists a \in A) b(x, \varepsilon) <_X a.\)

- \(\text{wb}(a) \overset{\text{def}}{=} \{b \in M_X \mid b <_X a\}.\)
Examples of located subsets

Proposition

- A splitting subset $V \subseteq M_X$ corresponds to a closed subset
  
  $$X_V \overset{\text{def}}{=} \{ x \in X \mid (\forall b(y, \delta) \in M_X) \, d(x, y) < \delta \rightarrow b(y, \delta) \in V \} .$$

  A closed subset $Y \subseteq X$ corresponds to a splitting subset
  
  $$V_Y \overset{\text{def}}{=} \{ b(x, \varepsilon) \in M_X \mid (\exists y \in Y) \, d(x, y) < \varepsilon \} .$$

  The correspondence is bijective.

- (Coquand, Palmgren, and Spitters (2011)) A splitting subset $V \subseteq M_X$ is located iff $X_V \subseteq X$ is semi-located, i.e. for each $x \in X$, the distance
  
  $$d(x, X_V) \overset{\text{def}}{=} \{ q \in \mathbb{Q}^>0 \mid (\exists y \in X_V) \, d(x, y) < q \}$$

  is a located upper real.
The space of located subsets
A **geometric theory** $T = (P, R)$ over a set $P$ of propositional symbols is a set $R$ of axioms of the form

$$p_0 \land \cdots \land p_{n-1} \vdash \bigvee_{i \in I} q_0^i \land \cdots \land q_{n_i-1}^i.$$

A **model** (ideal) of $T$ is subset $\alpha \subseteq P$ such that

$$\{p_0, \ldots, p_{n-1}\} \subseteq \alpha \implies (\exists i \in I) \{q_0^i, \ldots, q_{n_i-1}^i\} \subseteq \alpha$$

for all axioms $p_0 \land \cdots \land p_{n-1} \vdash \bigvee_{i \in I} q_0^i \land \cdots \land q_{n_i-1}^i$ in $T$.

**Problem.** Given a continuous cover $S$, find a geometric theory $T_L$ whose models are the located subsets of $S$. 
Example (Theory of splitting subsets)

Let \( S = (S, \triangleleft, \text{wb}) \) be a continuous cover.

Recall that \( V \subseteq S \) is splitting iff

1. \( a \triangleleft \{a_0, \ldots, a_{n-1}\} \text{ and } a \in V \implies (\exists i < n) a_i \in V, \)
2. \( a \in V \implies (\exists b \ll a) b \in V. \)

Thus, splitting subsets of \( S \) are the models of a geometric theory over \( S \) with the following axioms:

\[
a \vdash \bigvee_{b \ll a} b, \quad a \vdash \bigvee_{k<n} a_k \quad (a \triangleleft \{a_0, \ldots, a_{n-1}\})
\]
Example (Theory of splitting subsets)

Let $S = (S, \triangleleft, \text{wb})$ be a continuous cover.

Recall that $V \subseteq S$ is splitting iff

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Thus, splitting subsets of $S$ are the models of a geometric theory over $S$ with the following axioms:

$$a \vdash \bigvee_{b \ll a} b, \quad a \vdash \bigvee_{k < n} a_k \quad (a \triangleleft \{a_0, \ldots, a_{n-1}\})$$

Non-example (Located subsets)

A locatedness $a \ll b \implies a \notin V \lor b \in V$ is not geometric.

A naive approach requires non-geometric axiom:

$$\top \vdash (a \rightarrow \bot) \lor b \quad (a \ll b)$$

where $\top \overset{\text{def}}{=} \land \emptyset.$
Example (Theory of extended Dedekind reals)

- Consider $\mathcal{R}^u = (\mathbb{Q}, <_u, \text{wb})$ whose located subsets are the located (unbounded) upper reals.

- A located upper real is equivalent to an extended Dedekind real $(L, U)$, a pair of disjoint lower and upper reals that is located: $p < q \iff p \in L \lor q \in U$. 
Example (Theory of extended Dedekind reals)

Consider $\mathcal{R}^u = (\mathbb{Q}, \vartriangleleft_u, \text{wb})$ whose located subsets are the located (unbounded) upper reals.

A located upper real is equivalent to an extended Dedekind real $(L, U)$, a pair of disjoint lower and upper reals that is located:

$$p < q \implies p \in L \lor q \in U.$$ 

Extended Dedekind reals are the models of a theory $T_D$ over the propositional symbols $\{ (p, +\infty) \mid p \in \mathbb{Q} \} \cup \{ (\neg\infty, q) \mid q \in \mathbb{Q} \}$ with the following axioms:

$$\neg\infty, q \vdash \bigvee_{q' < q} (\neg\infty, q')$$

$$\neg\infty, q \vdash (\neg\infty, q') \quad (q < q')$$

Dual axioms for $(p, +\infty)$ ....

$$q, +\infty) \land (\neg\infty, q) \vdash \bot$$

$$\top \vdash (p, +\infty) \land (\neg\infty, q) \quad (p < q),$$

$$\top \overset{\text{def}}{=} \land \emptyset, \quad \bot \overset{\text{def}}{=} \lor \emptyset.$$
Let $S = (S, \triangleleft, \text{wb})$ be a continuous cover. A **cut** of $S$ is a pair $(L, U)$ of subsets of $S$ such that

1. $a \triangleleft \{a_0, \ldots, a_{n-1}\} \& a \in U \implies (\exists k < n) a_k \in U,$
2. $a \in U \implies (\exists b \ll a) b \in U,$
3. $a \triangleleft \{a_0, \ldots, a_{n-1}\} \& \{a_0, \ldots, a_{n-1}\} \subseteq L \implies a \in L,$
4. $a \in L \implies (\exists \{a_0, \ldots, a_{n-1}\} \gg a) \{a_0, \ldots, a_{n-1}\} \subseteq L,$
5. $L \cap U = \emptyset,$
6. $a \ll b \implies a \in L \lor b \in U.$
Let $S = (S, <, \text{wb})$ be a continuous cover. A **cut** of $S$ is a pair $(L, U)$ of subsets of $S$ such that

1. $a < \{a_0, \ldots, a_{n-1}\} \& a \in U \implies (\exists k < n) a_k \in U,$
2. $a \in U \implies (\exists b \ll a) b \in U,$
3. $a < \{a_0, \ldots, a_{n-1}\} \& \{a_0, \ldots, a_{n-1}\} \subseteq L \implies a \in L,$
4. $a \in L \implies (\exists \{a_0, \ldots, a_{n-1}\} \gg a) \{a_0, \ldots, a_{n-1}\} \subseteq L,$
5. $L \cap U = \emptyset,$
6. $a \ll b \implies a \in L \lor b \in U.$

**Proposition**

*There exists a bijective correspondence between the located subsets of $S$ and the cuts of $S$ given by*

$$V \mapsto (L_V, V),$$

$$L_V \overset{\text{def}}{=} \{a \in S \mid (\exists \{a_0, \ldots, a_{n-1}\} \gg a) (\forall k < n) a_k \notin V\}.$$
The theory of located subsets

Given a continuous cover $S$, define a geometric theory $T_L$ over a propositional symbols $P = \{l(a) \mid a \in S\} \cup \{u(a) \mid a \in S\}$ consisting of axioms:

\[
\begin{align*}
    u(a) & \vdash \bigvee_{k<n} u(a_k) & (a \lhd \{a_0, \ldots, a_{n-1}\}) \\
    u(a) & \vdash \bigvee_{b \ll a} u(b) \\
    l(a_0) \land \cdots \land l(a_{n-1}) & \vdash l(a) & (a \lhd \{a_0, \ldots, a_{n-1}\}) \\
    l(a) & \vdash \bigvee_{\{a_0, \ldots, a_{n-1}\} \gg a} l(a_0) \land \cdots \land l(a_{n-1}) \\
    l(a) \land u(a) & \vdash \bot \\
    \top & \vdash l(a) \lor u(b)
\end{align*}
\]

A model $\alpha \subseteq P$ corresponds to a cut of $S$ via

\[
\alpha \mapsto (\{a \mid l(a) \in \alpha\}, \{a \mid u(a) \in \alpha\}).
\]
Lawson topologies
A **formal topology** $S$ is a triple $S = (S, \triangleleft, \leq)$ where $(S, \leq)$ is a preorder and $\triangleleft \subseteq S \times \text{Pow}(S)$ is called a **cover** on $S$ such that

\[
\frac{a \in U}{\triangleleft U' }, \quad \frac{a \leq b}{\triangleleft b' }, \quad \frac{a \triangleleft U \quad U \triangleleft V}{\triangleleft V }, \quad \frac{a \triangleleft U \quad a \triangleleft V}{\triangleleft U \downarrow V },
\]

for all $a, b \in S$ and $U, V \subseteq S$ where

\[
U \downarrow V \overset{\text{def}}{=} \downarrow U \cap \downarrow V = \{ c \in S \mid (\exists a \in U) (\exists b \in V) c \leq a \land c \leq b \}.
\]

A geometric theory $T$ over propositional symbols $P$ determines a formal topology $S_T$ that corresponds to the frame presented by the theory $T$. 
Let \( S \) be a continuous cover, and let \( \mathcal{L}(S) \) be the formal topology associated with the geometric theory \( T_\mathcal{L} \); call \( \mathcal{L}(S) \) the space of located subsets of \( S \).

**Theory \( T_\mathcal{L} \)**

\[
\begin{align*}
\mathbf{u}(a) & \vdash \bigvee_{k<n} \mathbf{u}(a_k) \quad (a \ll \{a_0, \ldots, a_{n-1}\}) \\
\mathbf{u}(a) & \vdash \bigvee_{b\ll a} \mathbf{u}(b) \\
\mathbf{l}(a_0) \wedge \cdots \wedge \mathbf{l}(a_{n-1}) & \vdash \mathbf{l}(a) \quad (a \ll \{a_0, \ldots, a_{n-1}\}) \\
\mathbf{l}(a) & \vdash \bigvee_{\{a_0, \ldots, a_{n-1}\} \gg a} \mathbf{l}(a_0) \wedge \cdots \wedge \mathbf{l}(a_{n-1}) \\
\mathbf{l}(a) \wedge \mathbf{u}(a) & \vdash \bot \\
\top & \vdash \mathbf{l}(a) \vee \mathbf{u}(b) \quad (a \ll b)
\end{align*}
\]
The space of located subsets

Let $\mathcal{S}$ be a continuous cover, and let $\mathcal{L}(\mathcal{S})$ be the formal topology associated with the geometric theory $T_{\mathcal{L}}$; call $\mathcal{L}(\mathcal{S})$ the space of located subsets of $\mathcal{S}$.

Theory $T_{\mathcal{L}}$

\[
\mathbf{u}(a) \vdash \bigvee_{k<n} \mathbf{u}(a_k) \quad (a \triangleleft \{a_0, \ldots, a_{n-1}\})
\]

\[
\mathbf{u}(a) \vdash \bigvee_{b \ll a} \mathbf{u}(b)
\]

\[
\mathbf{l}(a_0) \land \cdots \land \mathbf{l}(a_{n-1}) \vdash \mathbf{l}(a) \quad (a \triangleleft \{a_0, \ldots, a_{n-1}\})
\]

\[
\mathbf{l}(a) \vdash \bigvee_{\{a_0, \ldots, a_{n-1}\} \gg a} \mathbf{l}(a_0) \land \cdots \land \mathbf{l}(a_{n-1})
\]

\[
\mathbf{l}(a) \land \mathbf{u}(a) \vdash \bot
\]

\[
\top \vdash \mathbf{l}(a) \lor \mathbf{u}(b)
\]
The space of located subsets

Let $S$ be a continuous cover, and let $\mathcal{L}(S)$ be the formal topology associated with the geometric theory $T_{\mathcal{L}}$; call $\mathcal{L}(S)$ the space of located subsets of $S$.

**Theory $T_{\mathcal{L}}$**

$$u(a) \vdash \bigvee_{k \leq n} u(a_k) \quad (a \ll \{a_0, \ldots, a_{n-1}\})$$

$$u(a) \vdash \bigvee_{b \ll a} u(b)$$

$$l(a_0) \land \cdots \land l(a_{n-1}) \vdash l(a) \quad (a \ll \{a_0, \ldots, a_{n-1}\})$$

$$l(a) \vdash \bigvee_{\{a_0, \ldots, a_{n-1}\} \gg a} l(a_0) \land \cdots \land l(a_{n-1})$$

$$l(a) \land u(a) \vdash \bot$$

$$\top \vdash l(a) \lor u(b) \quad (a \ll b)$$
The space of located subsets

Let $S$ be a continuous cover, and let $\mathcal{L}(S)$ be the formal topology associated with the geometric theory $T_\mathcal{L}$; call $\mathcal{L}(S)$ the space of located subsets of $S$.

Theory $T_\mathcal{L}$

\[ u(a) \vdash \bigvee_{k<n} u(a_k) \quad (a \prec \{a_0, \ldots, a_{n-1}\}) \]

\[ u(a) \vdash \bigvee_{b \ll a} u(b) \]

\[ l(a_0) \land \cdots \land l(a_{n-1}) \vdash l(a) \quad (a \prec \{a_0, \ldots, a_{n-1}\}) \]

\[ l(a) \vdash \bigvee_{\{a_0, \ldots, a_{n-1}\} \gg a} l(a_0) \land \cdots \land l(a_{n-1}) \]

\[ l(a) \land u(a) \vdash \bot \]

\[ \top \vdash l(a) \lor u(b) \quad (a \ll b) \]
A formal topology $S = (S, \triangleleft, \leq)$ is **regular** if

$$a \triangleleft \{b \in S \mid b \ll a\},$$

where $a \ll b \overset{\text{def}}{\iff} S \triangleleft a^* \cup \{b\}$ and $b^* \overset{\text{def}}{=} \{c \in S \mid b \downarrow c \ll \emptyset\}$.

A formal topology $S$ is **compact** if

$$S \triangleleft U \implies (\exists A \in \text{Fin}(U)) S \triangleleft A.$$

**Lemma (Johnstone (1982))**

*Every compact regular formal topology $S = (S, \triangleleft, \leq)$ is a continuous cover $(S, \triangleleft, \text{wb})$ with $\text{wb}(a) \overset{\text{def}}{=} \{b \in S \mid b \ll a\}$.***

**Lemma**

*Continuous maps between compact regular formal topologies are perfect. Hence, the category $\textbf{KReg}$ of compact regular formal topologies and continuous maps is a full subcategory of $\textbf{CCov}$.***
Proposition

1. $\mathcal{L}(S)$ is a compact regular formal topology.

2. There exists a perfect map $\iota_S : \mathcal{L}(S) \to S$ such that for any compact regular formal topology $S'$ and a perfect map $r : S' \to S$, there exists a unique continuous map $\tilde{r} : S' \to \mathcal{L}(S)$ such that

\[
\begin{array}{ccc}
\mathcal{L}(S) & \xleftarrow{\exists! \tilde{r}} & S' \\
\downarrow \iota_S & & \downarrow r \\
S & \xleftarrow{r} & \\
\end{array}
\]

Theorem

The construction $\mathcal{L}(S)$ is the right adjoint to the forgetful functor $K\text{Reg} \to \text{CCov}$. 
Lawson topology

Classically, the right adjoint to the forgetful functor $\text{KReg} \to \text{CCov}$ is given by the Lawson topologies on continuous lattices.

**Theorem.** $\mathcal{L}(S)$ represents the Lawson topology on $\text{Sat}(S)$.

**Theorem.** The monad $K_L$ on $\text{KReg}$ induced by the adjunction is naturally isomorphic to the Vietoris monad on $\text{KReg}$.

\[ \Diamond a \vdash \bigvee_{k < n} \Diamond a_k \quad (a \triangleleft \{a_0, \ldots, a_{n-1}\}) \]

\[ \Diamond a \vdash \bigvee_{b \ll a} \Diamond b \]

\[ \top \vdash \bigvee \{\Box A \mid A \in \text{Fin} S\} \]

\[ \Box A \vdash \Box B \quad (A \triangleleft B) \]

\[ \Box A \land \Box B \vdash \bigvee \{\Box C \mid C \ll A \land C \ll B\} \]

\[ \Box A \land \Diamond a \vdash \bigvee \{\Diamond b \mid b \in A \downarrow a\} \]

\[ \Box (A \cup \{a\}) \vdash \Box A \lor \Diamond a. \]
Lawson topology

Classically, the right adjoint to the forgetful functor $\textbf{KReg} \to \textbf{CCov}$ is given by the Lawson topologies on continuous lattices.

**Theorem.** $\mathcal{L}(S)$ represents the Lawson topology on $\text{Sat}(S)$.

**Theorem.** The monad $K_{\mathcal{L}}$ on $\textbf{KReg}$ induced by the adjunction is naturally isomorphic to the Vietoris monad on $\textbf{KReg}$.

\[
\diamond a \vdash \bigvee_{k<n} \diamond a_k \quad \quad (a \rhd \{a_0, \ldots, a_{n-1}\})
\]

\[
\diamond a \vdash \bigvee_{b \ll a} \diamond b
\]

\[
\top \vdash \bigvee \{\square A \mid A \in \text{Fin} S\}
\]

\[
\square A \vdash \square B \quad \quad (A \rhd B)
\]

\[
\square A \land \square B \vdash \bigvee \{\square C \mid C \ll A \land C \ll B\}
\]

\[
\square A \land \diamond a \vdash \bigvee \{\diamond b \mid b \in A \downarrow a\}
\]

\[
\square (A \cup \{a\}) \vdash \square A \lor \diamond a.
\]
Lawson topology

Classically, the right adjoint to the forgetful functor $\text{KReg} \to \text{CCov}$ is given by the Lawson topologies on continuous lattices.

**Theorem.** $\mathcal{L}(S)$ represents the Lawson topology on $\text{Sat}(S)$.

**Theorem.** The monad $K_{\mathcal{L}}$ on $\text{KReg}$ induced by the adjunction is naturally isomorphic to the Vietoris monad on $\text{KReg}$.

\[
\diamond a \vdash \bigvee_{k<n} \diamond a_k \\
\diamond a \vdash \bigvee_{b \ll a} \diamond b \\
\top \vdash \bigvee \{\square A \mid A \in \text{Fin } S\} \\
\square A \vdash \square B \\
\square A \land \square B \vdash \bigvee \{\square C \mid C \ll A \land C \ll B\} \\
\square A \land \diamond a \vdash \bigvee \{\diamond b \mid b \in A \downarrow a\} \\
\square (A \cup \{a\}) \vdash \square A \lor \diamond a.
\]
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\[
\Diamond a \vdash \bigvee_{k<n} \Diamond a_k \quad (a \triangleleft \{a_0, \ldots, a_{n-1}\})
\]

\[
\Diamond a \vdash \bigvee_{b \ll a} \Diamond b
\]

\[
\top \vdash \bigvee \{\Box A \mid A \in \text{Fin} \, S\}
\]

\[
\Box A \vdash \Box B \quad (A \triangleleft B)
\]

\[
\Box A \land \Box B \vdash \bigvee \{\Box C \mid C \ll A \land C \ll B\}
\]

\[
\Box A \land \Diamond a \vdash \bigvee \{\Diamond b \mid b \in A \downarrow a\}
\]

\[
\Box (A \cup \{a\}) \vdash \Box A \lor \Diamond a.
\]
The notion located subset for continuous cover captures well-known examples of located subsets.

Located subsets can be characterised geometrically by the notion of cuts.

The space $\mathcal{L}(S)$ of located subsets of a continuous cover $S$ is the Lawson topology on $S$.

The monad on $\textbf{KReg}$ induced by the construction $\mathcal{L}(-)$ is the Vietoris monad on $\textbf{KReg}$. 


