States of free product algebras

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(joint work with Tommaso Flaminio and Lluis Godo)

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Background

In Hájek's setting of mathematical fuzzy logic, BL (Basic Logic) plays a fundamental role, as the logic of all continuous t-norms.

Its algebraic semantics, the variety of BL-algebras, is generated by the class of BL-algebras on $[0, 1]$, which are defined by a continuous t-norm and its residuum.

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Three prominent axiomatic extensions, with corresponding algebraic semantics:

- Łukasiewicz logic (involutive), MV-algebras
- Gödel logic (contractive), Gödel algebras
- Product logic (cancellative), Product algebras

[Mostert-Shields Thm.]: a t-norm is continuous if and only if it can be built from the previous three ones by the construction of ordinal sum.

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A BL-algebra is a (commutative, integral, pointed, bounded) residuated lattice $\mathbf{A} = (A, \odot, \to, \wedge, \vee, 0, 1)$ which satisfies the following equations:

$$\begin{array}{lcll} (x \to y) \vee (y \to x) & = & 1 & \text{(prelinearity)}, \\ x \odot (x \to y) & = & x \wedge y & \text{(divisibility)}. \end{array}$$

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An MV-algebra $\mathbf{A}$ is an involutive BL-algebra: $\neg\neg x = x$.

A Gödel algebra $\mathbf{A}$ is a contractive BL-algebra: $x \cdot x = x$.

A product algebra $\mathbf{A}$ is a BL-algebra that satisfies: $x \wedge \neg x = 0$ and $\neg\neg x \to ((y \odot x \to z \odot x) \to (y \to z)) = 1$.

In what follows: $\neg x := x \to 0$, $x^n := x \odot \ldots \odot x$ (n-times), $x \oplus y = \neg(\neg x \odot \neg y)$.

Background

Standard MV-algebra: $[\mathbf{0},\mathbf{1}]_{\text{Ł}} = ([0,1], \odot_{\text{Ł}}, \to_{\text{Ł}}, \min, \max, 0, 1)$

$$\begin{array}{rcl} x \odot_{\text{Ł}} y & = & \max\{0, x+y-1\} \\ x \to_{\text{Ł}} y & = & 1 \text{ if } x \leq y, \\ & & 1-x+y \text{ otherwise.} \end{array}$$

Standard Gödel algebra: $[\mathbf{0},\mathbf{1}]_{\mathbf{G}} = ([0,1], \odot_G, \to_G, \min, \max, 0, 1)$

$$\begin{array}{rcl} x \odot_G y & = & \min\{x, y\} \\ x \to_G y & = & 1 \text{ if } x \leq y, \\ & & y \text{ otherwise.} \end{array}$$

Standard product algebra: $[\mathbf{0},\mathbf{1}]_{\mathbf{P}} = ([0,1], \odot_P, \to_P, \min, \max, 0, 1)$

$$\begin{array}{rcl} x \odot_P y & = & x \cdot y \\ x \to_P y & = & 1 \text{ if } x \leq y, \\ & & y/x \text{ otherwise.} \end{array}$$

Free algebras

For L any of MV, Gödel and product logics, and $\mathbb{L}$ its algebraic semantics, let $\mathcal{F}_{\mathbb{L}}(n)$ be the free $\mathbb{L}$-algebra over n generators, i.e. the Lindenbaum algebra of L-logic over n variables.

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Since $[0,1]_L$ is generic for the variety, $\mathcal{F}_{\mathbb{L}}(n)$ is, up to isomorphisms, the subalgebra of all functions $[0,1]^n \to [0,1]$ generated by the projection maps $\pi_1, \ldots, \pi_n : [0,1]^n \to [0,1]$, with operations defined componentwise by the standard ones.

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Thus, every element of $f \in \mathcal{F}_{\mathbb{L}}(n)$ can be regarded as a function $f : [0,1]^n \to [0,1]$. For example, for product logic:

formula	product function
$\varphi = p \odot (q \wedge r)$	$f_\varphi(x,y,z) = x \cdot \min(y,z)$

States of MV and Gödel algebras

[Mundici, 95]: Given any MV-algebra $\mathbf{A} = (A, \odot, \to, \wedge, \vee, 0, 1)$, a state of $\mathbf{A}$ is a map $s : A \to [0, 1]$ such that:

(I) $s(1) = 1$,

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[Aguzzoli-Gerla-Marra, 2008]: Let $\mathcal{F}_{\mathbb{G}}(n)$ be the free Gödel algebra on n generators. A state of $\mathcal{F}_{\mathbb{G}}(n)$ is a map $s : \mathcal{F}_{\mathbb{G}}(n) \to [0, 1]$ such that:

(I) $s(0) = 0$ and $s(1) = 1$,

(II) $f \leq g$ implies $s(f) \leq s(g)$

(III) $s(f \vee g) = s(f) + s(g) - s(f \wedge g)$

(IV) if f, g, h are either join-irreducible elements or equal to 0, and satisfy $f < g < h$, then $s(f) = s(g)$ implies $s(g) = s(h)$.

Integral representation

Let $\mathcal{F}(n)$ be the free Gödel (or MV) algebra on n generators. Let s be a state on $\mathcal{F}(n)$. Then there exists a (unique) regular Borel probability measure μ on $[0,1]^n$ such that, for any $f \in \mathcal{F}(n)$,

$$s(f) = \int_{[0,1]^n} f \mathrm{d}\mu.$$

For MV-algebras, Kroupa-Panti Theorem ['06 - '09] establishes an integral representation theorem for states of any MV-algebra.

States of product logic

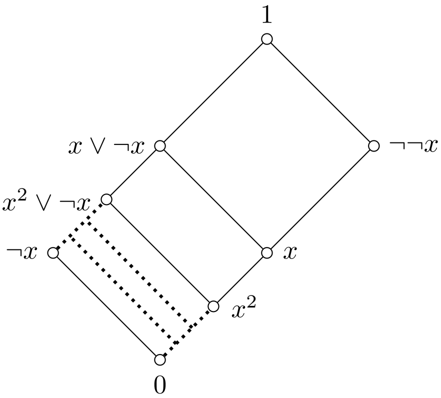
Our aim is to introduce and study states for product logic, the remaining fundamental many-valued logic for which such a notion is still lacking.

In particular, we will study states of $\mathcal{F}_{\mathbb{P}}(n)$, the free product algebra over n generators, i.e. the Lindenbaum algebra of product logic over n variables.

Since every element of $f \in \mathcal{F}_{\mathbb{P}}(n)$ can be regarded as a function $f : [0,1]^n \to [0,1]$ we will refer to them as *product functions*.

$\mathcal{F}_{\mathbb{P}}(1)$

The lattice of the free Product algebra with one generator $\mathcal{F}_{\mathbb{P}}(1)$:



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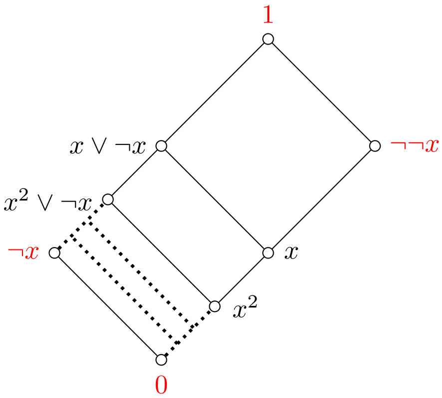

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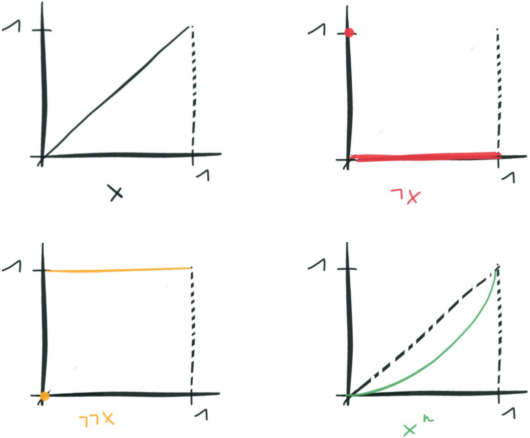

Notice: product functions are not continuous;
the Boolean atoms $\neg x$ and $\neg\neg x$ determine a partition of the domain, given by the areas where they assume value 0 or 1.

Free product algebras

In the following, we will denote with:

- p_ϵ, with $\epsilon \in \Sigma$, the Boolean atoms of $\mathcal{F}_{\mathbb{P}}(n)$;
- G_ϵ the part of the domain where p_ϵ has value 1 and 0 outside. The G_ϵ's, with $\epsilon \in \Sigma$, form a partition of $[0,1]^n$.

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Product functions are not continuous, like the fuctions of the free MV-algebra, nor in a finite number, as the functions of the free n-generated Gödel algebra. But:

Fact

Every product function $f : [0,1]^n \to [0,1]$ is such that, for every $\epsilon \in \Sigma$, its restriction f_ϵ to G_ϵ is continuous.

In fact, f_ϵ is either 0 or a piecewise monomial function (i.e. $g(x_1, \dots x_n) = 1 \wedge x_1^{m_1} \dots x_n^{m_n}$, with $m_i \in \mathbb{Z}$) [Cintula, Gerla].

States of $\mathcal{F}_{\mathbb{P}}(n)$

Definition

A state of $\mathcal{F}_{\mathbb{P}}(n)$ is a map $s : \mathcal{F}_{\mathbb{P}}(n) \to [0, 1]$ satisfying the following conditions:

S1. $s(1) = 1$ and $s(0) = 0$,

S2. $s(f \wedge g) + s(f \vee g) = s(f) + s(g)$,

S3. If $f \leq g$, then $s(f) \leq s(g)$,

S4. If $f \neq 0$, then $s(f) = 0$ implies $s(\neg\neg f) = 0$.

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S4. If $f \neq 0$, then $s(f) = 0$ implies $s(\neg\neg f) = 0$.

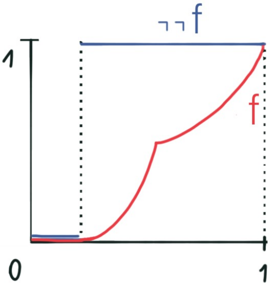
Notice that:

S2: a state is a Birkhoff's lattice valuation

S4: only property (indirectly) involving the monoidal operation

Condition S4

Recall: $\neg\neg f(x) = \begin{cases} 1, & \text{if } f(x) > 0 \\ 0, & \text{if } f(x) = 0 \end{cases}$,



$s(f) = 0$ implies $s(\neg\neg f) = 0$.

States on $\mathcal{F}_{\mathbb{P}}(n)$: some properties

For any state $s : \mathcal{F}_{\mathbb{P}}(n) \to [0, 1]$ the following properties hold:

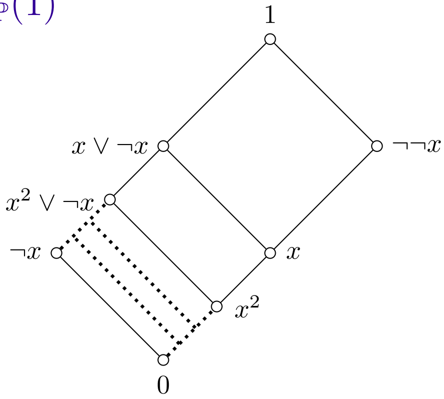
(I) s restricted to $\mathscr{B}(\mathcal{F}_{\mathbb{P}}(n))$ is a (finitely additive) probability measure

(II) $s(f \vee \neg f) = s(f) + s(\neg f)$

(III) $s(f \leftrightarrow g) = s(f \to g) + s(g \to f) - 1$

(IV) $s(\neg f) + s(\neg\neg f) = 1$

States on $\mathcal{F}_{\mathbb{P}}(1)$



- if $s(\neg x) = \alpha$, and $s(\neg\neg x) = \beta$ then $\alpha + \beta = 1$;
- either $s(\neg\neg x) = s(x) = s(x^n) = 0$ for all n, or all of them are positive;
- $s(x^n) \leq s(x^m)$, whenever $n \geq m$;
- $s(x^n \vee \neg x) = s(x^n) + s(\neg x)$.

Towards an integral representation

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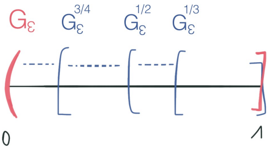

$$G_\epsilon = \prod_{i=1}^{n} B_i, \text{ where } B_i = (0,1] \text{ or } B_i = \{0\}$$

$$G_\epsilon^q = \prod_{i=1}^{n} B_i^q, \text{ where } B_i = [q,1] \text{ or } B_i = \{0\}$$

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Tool:

Theorem (Riesz representation theorem)

Let X be a locally compact Hausdorff space, and let $\sigma : \mathscr{C}(X) \to \mathbb{R}$ be a positive linear functional on the space $\mathscr{C}(X)$ of continuous functions with compact support. Then there is a unique regular Borel measure μ on X such that

$$\sigma(f) = \int_X f \mathrm{d}\mu$$

for each $f \in \mathscr{C}(X)$.

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Given a state $s : \mathcal{F}_{\mathbb{P}}(n) \to [0,1]$, define $s_\epsilon : \mathcal{F}_{\mathbb{P}}(n)_{|G_\epsilon} \to [0,1]$ by

$$s_\epsilon(g_\epsilon) = \frac{s(g \wedge p_\epsilon)}{s(p_\epsilon)}$$

(1) for each $q \in [0,1]_Q$, consider its induced map s_ϵ^q's on product functions restricted to the G_ϵ^q's

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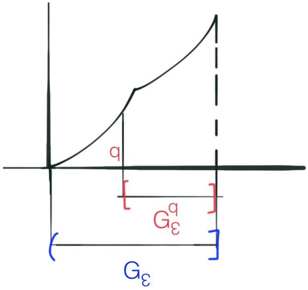

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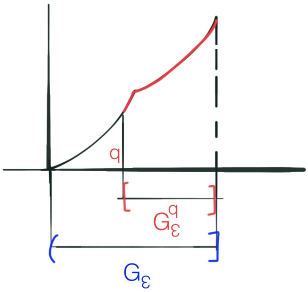

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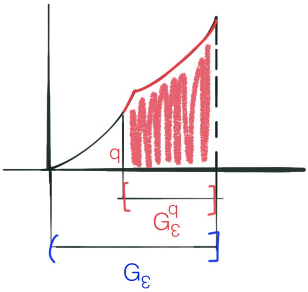

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Continuous functions on G_ϵ^q can be uniformly approximated by linear combinations of the functions of $\mathcal{F}_{\mathbb{P}}(n)$ restricted to such subsets:

(2) extend s_ϵ^q to a monotone linear functional τ_ϵ^q on the linear span Λ_ϵ^q of $\mathcal{F}_{\mathbb{P}}(n)$ over G_ϵ^q.

(3) uniformly approximate continuous functions $\mathscr{C}(G_\epsilon^q)$ by sequences in Λ_ϵ^q

(4) suitably extend τ_ϵ^q to a linear functional on $\mathscr{C}(G_\epsilon^q)$

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(4) suitably extend τ^q_ϵ to a linear functional on $\mathscr{C}(G^q_\epsilon)$

(5) apply Riesz theorem at the level of G^q_ϵ and get a unique Borel probability measure μ^q_ϵ representing τ^q_ϵ

Integral representation for states on product functions

Now:

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Theorem

For every state s of $\mathcal{F}_{\mathbb{P}}(n)$ there is a unique regular Borel probability measure μ on $[0,1]^n$, such that for every $f \in \mathcal{F}_{\mathbb{P}}(n)$:

$$s(f) = \int_{[0,1]^n} f \, \mathrm{d}\mu.$$

State space and its extremal points

$\mathcal{S}(n)$: set of all states of $\mathcal{F}_{\mathbb{P}}(n)$
$\mathcal{H}(n)$: set of product logic homomorphisms of $\mathcal{F}_{\mathbb{P}}(n)$ into $[0,1]_P$
$\mathcal{M}(n)$: set of all regular Borel probability measures on $[0,1]^n$

The state space $\mathcal{S}(n)$ results to be a closed convex subset of $[0,1]^{\mathcal{F}_{\mathbb{P}}(n)}$. Moreover, the map $\delta : \mathcal{S}(n) \to \mathcal{M}(n)$ such that $\delta(s) = \mu$ is bijective and affine.

Theorem

The following are equivalent for a state $s : \mathcal{F}_{\mathbb{P}}(n) \to [0,1]$:

(i) s *is extremal;*

(ii) $\delta(s)$ *is a Dirac measure;*

(iii) $s \in \mathcal{H}(n)$.

State space and its extremal points

Thus, via Krein-Milman Theorem we obtain the following:

Corollary

For every $n \in \mathbb{N}$, the state space $\mathcal{S}(n)$ is the convex closure of the set of product homomorphisms from $\mathcal{F}_{\mathbb{P}}(n)$ into $[0,1]_P$.

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Close analogy with MV and Gödel.

MV: The state space of an MV-algebra **A** is a compact convex space generated by its extremal states, that coincide with the homomorphisms of **A** to $[0,1]_{\text{Ł}}$.

Gödel: States of $\mathcal{F}_{\mathbb{G}}(n)$ are precisely the convex combinations of finitely many truth value assignments.

Conclusions and future work

- Our axiomatization of states characterizes Lebesgue integral of the functions belonging to the free n-generated product algebra with respect to Borel probability measures on $[0,1]^n$.

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- States of (free) BL-algebras?