

# INFINITARY PROPOSITIONAL LOGICS AND SUBDIRECT REPRESENTATION

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Topology, Algebra, and Categories in Logic

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# INTRODUCTION

## THEOREM (BIRKHOFF'S SUBDIRECT REPRESENTATION)

*Let  $\mathbb{V}$  be a variety and  $\mathbb{Q}$  a quasivariety, then*

$$\mathbb{V} = \mathbf{P}_{\mathbf{SD}}(\mathbb{V}_{\mathbf{SI}}) \quad \text{and} \quad \mathbb{Q} = \mathbf{P}_{\mathbf{SD}}(\mathbb{Q}_{\mathbf{RSI}})$$

$\mathbb{V}_{\mathbf{SI}}$  ... subdirectly irreducible algebras in  $\mathbb{V}$

$\mathbb{Q}_{\mathbf{RSI}}$  ... relatively subdirectly irreducible algebras in  $\mathbb{Q}$

$\mathbf{P}_{\mathbf{SD}}$  ... operator for subdirect products

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$\mathbf{P}_{\mathbf{SD}}$  ... operator for subdirect products

In abstract algebraic logic this amounts to

## THEOREM

If  $L$  is a *finitary* logic then

$$\mathbf{MOD}^*(L) = \mathbf{P}_{\mathbf{SD}}(\mathbf{MOD}^*(L)_{\mathbf{RSI}})$$

## PRELIMINARIES: LOGIC

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A **logic**  $L$  is a relation between sets of formulas and formulas, satisfying

We write  $\Gamma \vdash_L \varphi$  instead of  $\langle \Gamma, \varphi \rangle \in L$

$\varphi \vdash_L \varphi$  (reflexivity)

$\Gamma \vdash_L \varphi \Rightarrow \Gamma, \Delta \vdash_L \varphi$  (monotonicity)

$\Gamma \vdash_L \Delta$  and  $\Delta \vdash_L \varphi \Rightarrow \Gamma \vdash_L \varphi$  (cut)

$\Gamma \vdash_L \varphi \Rightarrow \sigma\Gamma \vdash_L \sigma\varphi$  for each substitution  $\sigma$  (structurality)

$L$  is **finitary** if whenever  $\Gamma \vdash_L \varphi$  then  $\Gamma' \vdash_L \varphi$  for a finite  $\Gamma' \subseteq \Gamma$

## PRELIMINARIES: SEMANTICS

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the corresponding **semantical consequence relation** is:

$$\Gamma \models_{\mathbb{K}} \varphi \iff \text{for every } \langle A, F \rangle \in \mathbb{K}, \text{ and } v \in \text{Hom}(\mathbf{Fm}_{\mathcal{L}}, A) \\ \text{if } v[\Gamma] \subseteq F \text{ then } v(\varphi) \in F$$



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$\models_{\mathbb{K}}$  is a logic.

A matrix  $A$  is a **model** of  $L$ ,  $A \in \mathbf{MOD}(L)$ , if  $\vdash_L \subseteq \models_A$

$$\Gamma \vdash_L \varphi \iff \Gamma \models_{\mathbf{MOD}(L)} \varphi \quad (\text{completeness})$$

## PRELIMINARIES: MODELS

Let  $L$  be a logic.

$\mathbf{MOD}^*(L) = \{\langle A, F \rangle \in \mathbf{MOD}(L) : \langle A, F \rangle \text{ is reduced } (\Omega_A(F) = \text{Id}_A)\}$

$\mathbf{MOD}^*(L)_{\text{RFSI}} = \{A \in \mathbf{MOD}^*(L) : \\ \text{is finitely subdirectly irreducible relative to } \mathbf{MOD}^*(L)\}$

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$L$  is called **R(F)SI-complete** when

$$\Gamma \vdash_L \varphi \iff \Gamma \models_{\mathbf{MOD}^*(L)_{\text{R(F)SI}}} \varphi$$

$L$  is called **(finitely) subdirectly representable** when

$$\mathbf{MOD}^*(L) = \mathbf{P}_{\text{SD}}(\mathbf{MOD}^*(L)_{\text{R(F)SI}})$$

# IPEP AND CIPEP

Closure systems in AAL: Let  $L$  be a logic.

1. For every algebra  $A$ , the closure system of all  $L$ -filters is

$$\mathcal{F}i_L(A) = \{F \subseteq A : \langle A, F \rangle \in \mathbf{MOD}(L)\}$$

2.  $\mathbf{Th}(L)$  is the closure system of all  $L$ -theories (deductively closed sets of formulas).

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Let  $\mathcal{C}$  be a closure system on a set  $A$ .

- ▶  $\mathcal{B} \subseteq \mathcal{C}$  is a **basis** of  $\mathcal{C}$  when every  $X \in \mathcal{C}$  is an intersection of members from  $\mathcal{B}$ .
- ▶  $X \in \mathcal{C}$  is **(completely)  $\cap$ -prime** if it is (completely)  $\cap$ -irreducible in  $\mathcal{C}$ .
- ▶  $\mathcal{C}$  has the **(completely) intersection-prime extension property**, **(C)IPEP**, if the (completely)  $\cap$ -prime members form a basis of  $\mathcal{C}$ .

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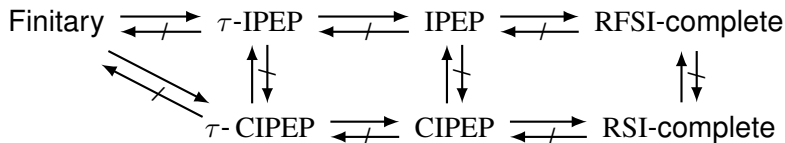
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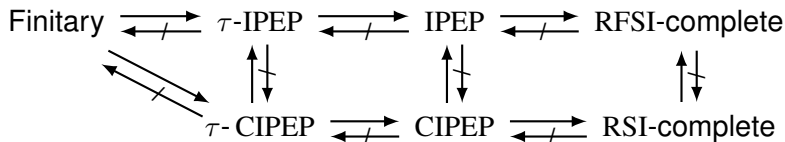
$L$  **has (C)IPEP** if  $\mathbf{Th}(L)$  does.

$L$  **has the transferred (C)IPEP**,  $\tau$ -(C)IPEP, if  $\mathcal{F}i_L(A)$  has (C)IPEP for every algebra  $A$ .

# HIERARCHY OF INFINITARY LOGICS



# $\tau$ -(C)IPEP CHARACTERIZATION



## THEOREM

For every protoalgebraic logic  $L$ , the following are equivalent:

- ▶  $L$  is (finitely) subdirectly representable,  
 $(\mathbf{MOD}^*(L) = \mathbf{P}_{\mathbf{SD}}(\mathbf{MOD}^*(L)_{\mathbf{R(F)SI}}))$
- ▶  $L$  has the  $\tau$ -(C)IPEP,
- ▶  $\mathcal{Fi}_L(\mathbf{Fm}_{\mathcal{L}}(\kappa))$  has the (C)IPEP for every cardinal  $\kappa$ .



## EXAMPLES:PRODUCT LOGIC

Language  $\{\rightarrow, \&, \bar{0}\}$  with *countable* set *Var*

Let  $[0, 1]_{\Pi}$  be the standard product algebra

- ▶ Universe  $[0, 1]$  of reals,
- ▶  $a \rightarrow^{[0,1]_{\Pi}} b = \min\{1, b/a\}$
- ▶  $a \&^{[0,1]_{\Pi}} b = a \cdot b$ , and  $\bar{0}^{[0,1]_{\Pi}} = 0$ .

The **infinitary product logic**,  $\Pi_{\infty}$ , is semantically given by the matrix  $\langle [0, 1]_{\Pi}, \{1\} \rangle$ .

$$\Gamma \vdash_{\Pi_{\infty}} \varphi \iff \begin{array}{l} \text{for every } v \in \text{Hom}(\mathbf{Fm}_{\mathcal{L}}, [0, 1]_{\Pi}) \\ \text{if } v[\Gamma] \subseteq \{1\} \text{ then } v(\varphi) = 1 \end{array}$$

## EXAMPLES: ŁUKASIEWICZ LOGIC

Language  $\{\rightarrow, \&, \bar{0}\}$  with *countable* set  $Var$ .

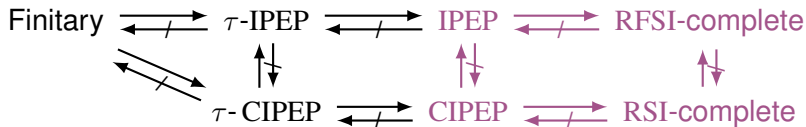
Let  $[0, 1]_{\mathbb{L}}$  be the standard Łukasiewicz algebra:

- ▶ Universe  $[0, 1]$  of reals,
- ▶  $a \rightarrow^{[0,1]_{\mathbb{L}}} b = \min\{1 - a + b, 1\}$
- ▶  $a \&^{[0,1]_{\mathbb{L}}} b = \max\{a + b - 1, 0\}$ , and  $\bar{0}^{[0,1]_{\mathbb{L}}} = 0$ .

The **infinitary Łukasiewicz logic**,  $\mathbb{L}_{\infty}$ , is semantically given by the matrix  $\langle [0, 1]_{\mathbb{L}}, \{1\} \rangle$ .

$$\Gamma \vdash_{\mathbb{L}_{\infty}} \varphi \iff \begin{array}{l} \text{for every } v \in \text{Hom}(\mathbf{Fm}_{\mathcal{L}}, [0, 1]_{\mathbb{L}}) \\ \text{if } v[\Gamma] \subseteq \{1\} \text{ then } v(\varphi) = 1 \end{array}$$

# PROPERTIES: INFINITARY PRODUCT LOGIC

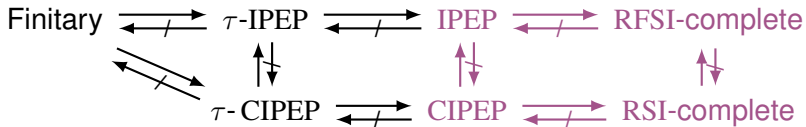


## THEOREM

*The logic  $\Pi_\infty$  is not even finitely subdirectly representable (equivalently it does not have  $\tau$ -IPEP). That is*

$$\mathbf{MOD}^*(\Pi_\infty) \neq \mathbf{P}_{\text{SD}}(\mathbf{MOD}^*(\Pi_\infty)_{\text{RFSI}}).$$

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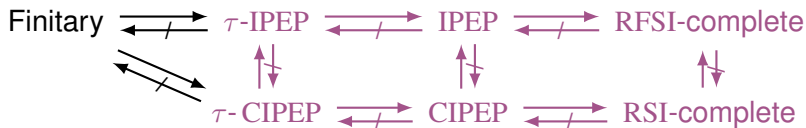
$$\mathbf{MOD}^*(\Pi_\infty) \neq \mathbf{P}_{\mathbf{SD}}(\mathbf{MOD}^*(\Pi_\infty)_{\mathbf{RFSI}}).$$

## COROLLARY

For its equivalent algebraic semantics,  $\mathbf{ALG}^*(\Pi_\infty)$ , we obtain

$$\mathbf{ALG}^*(\Pi_\infty) \neq \mathbf{P}_{\mathbf{SD}}(\mathbf{ALG}^*(\Pi_\infty)_{\mathbf{R(F)SI}}).$$

# PROPERTIES: INFINITARY ŁUKASIEWICZ LOGIC



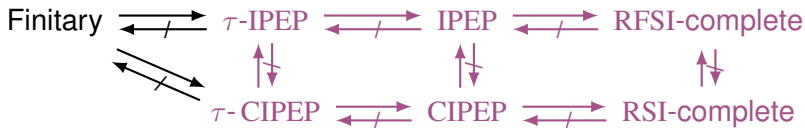
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$\mathbb{L}_\infty$  has the  $\tau$ -CIPEP and is subdirectly representable, that is

$$\mathbf{MOD}^*(\mathbb{L}_\infty) = \mathbf{P}_{\mathbf{SD}}(\mathbf{MOD}^*(\mathbb{L}_\infty)_{\mathbf{RSI}})$$

in particular, it is representable by chains.

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## COROLLARY

The equivalent algebraic semantics of  $\mathbb{L}_\infty$ ,  $\mathbf{ALG}^*(\mathbb{L}_\infty)$ , is not a quasivariety and yet

$$\mathbf{ALG}^*(\mathbb{L}_\infty) = \mathbf{P}_{\mathbf{SD}}(\mathbf{ALG}^*(\mathbb{L}_\infty)_{\mathbf{RSI}}).$$

# LOGIC OF THE STANDARD MV-CHAIN: CARDINALITY

## LEMMA

*The logic of  $[0, 1]_{\mathbb{L}}$  in  $\kappa$ -many variables,  $\vdash$ , has cardinality at most  $\aleph_1$ .*

## PROOF.

Assume  $\Gamma \vdash \varphi$ . We find a countable  $\Gamma' \subseteq \Gamma$  such that  $\Gamma' \vdash \varphi$ :

Consider the (compact) product topology  $[0, 1]^\kappa$  and define sets

$$\text{SAT}(\varphi) = \{v \in [0, 1]^\kappa : v(\varphi) = 1\}$$

$$\text{NSAT}(\varphi) = \{v \in [0, 1]^\kappa : v(\varphi) \neq 1\}$$

$$\text{SAT}_q(\varphi) = \{v \in [0, 1]^\kappa : v(\varphi) > q\}$$

$$\Gamma \vdash \varphi \iff \bigcup_{\gamma \in \Gamma} \text{NSAT}(\gamma) \cup \text{SAT}(\varphi) = [0, 1]^\kappa, \quad (*)$$

The connectives of Łukasiewicz logic are continuous, thus  $\text{NSAT}(\varphi)$  and  $\text{SAT}_q(\varphi)$  are open sets in  $[0, 1]^\kappa$  ( $\varphi : [0, 1]^\kappa \rightarrow [0, 1]$  s.t.  $v \mapsto v(\varphi)$ ).

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## PROOF.

$$\bigcup_{\gamma \in \Gamma} \text{NSAT}(\gamma) \cup \text{SAT}(\varphi) = [0, 1]^{\kappa}, \quad (1)$$

since  $\text{SAT}(\varphi) \subseteq \text{SAT}_q(\varphi)$ , we get

$$\bigcup_{\gamma \in \Gamma} \text{NSAT}(\gamma) \cup \text{SAT}_q(\varphi) = [0, 1]^{\kappa}, \quad (2)$$

Let  $\Gamma_q \subseteq \Gamma$  generate finite subcover of (2) and set  $\Gamma' = \bigcup_{q \in (0,1)} \Gamma_q$ .

$$\{1\} = \bigcap_{q \in (0,1)} \uparrow q \text{ implies } \bigcup_{\gamma \in \Gamma'} \text{NSAT}(\gamma) \cup \text{SAT}(\varphi) = [0, 1]^{\kappa},$$

consequently, by (\*),  $\Gamma' \vdash \varphi$ . □



# LOGIC WITH $\tau$ -IPEP WHICH IS NOT RSI-COMPLETE

$\mathcal{L} = \{\rightarrow, \&, \bar{0}\} \cup \{\bar{q} : q \in (0, 1] \cap \mathbb{Q}\}$  and  $Var = \omega$ .

$[0, 1]_{\mathbb{L}}^{\mathbb{Q}}$  is  $[0, 1]_{\mathbb{L}}$  with naturally defined rational constants.

Let  $\mathbf{L}$  be the logic preserving degrees of truth in  $[0, 1]_{\mathbb{L}}^{\mathbb{Q}}$ , i.e.

$$\Gamma \vdash_{\mathbf{L}} \varphi \iff \bigwedge v[\Gamma] \leq v(\varphi), \text{ for all } v \in \text{Hom}(\mathbf{Fm}_{\mathcal{L}}, [0, 1]_{\mathbb{L}}^{\mathbb{Q}}).$$

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- ▶  $\mathbf{L}$  is equivalential (with implication  $x \Rightarrow y = \{(x \rightarrow y)^n : n \in \omega\}$ ), but not algebraizable.
- ▶  $\mathbf{L}$  has the  $\tau$ -IPEP (and is finitely subdirectly representable).
- ▶  $\mathbf{L}$  has no RSI-models, thus it is not RSI-complete.

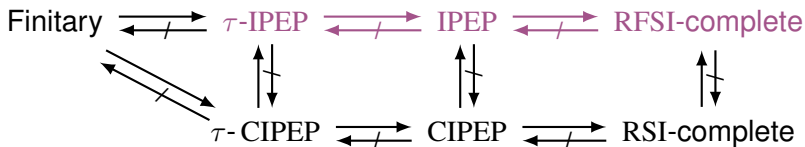
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Thank you,  
Enjoy Prague!

# CARDINALITY UPPER BOUND

## THEOREM

*Suppose  $\lambda$  is a regular cardinal and  $\mathbb{K}$  is a class of matrices, such that  $|\mathbb{K}| < \lambda$ . Further suppose that for every  $\langle A, F \rangle \in \mathbb{K}$ :*

- 1. There is a compact topology  $\tau$  on  $A$  such that all of the connectives are continuous w.r.t.  $\tau$ ,*
- 2.  $F$  can be written as an intersection of strictly less  $\lambda$  open sets in  $\tau$ ,*
- 3.  $A \setminus F \in \tau$ ,*

*then  $L_{\mathbb{K}, \kappa}$  has cardinality at most  $\lambda$  for every infinite  $\kappa$ .*