Infinitary Propositional Logics and Subdirect Representation

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Topology, Algebra, and Categories in Logic

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INTRODUCTION

THEOREM (BIRKHOFF'S SUBDIRECT REPRESENTATION)

Let $\mathbb{V}$ be a variety and $\mathbb{Q}$ a quasivariety, then

$$\mathbb{V} = \mathbf{P_{SD}}(\mathbb{V}_{\mathrm{SI}}) \quad \textit{and} \quad \mathbb{Q} = \mathbf{P_{SD}}(\mathbb{Q}_{\mathrm{RSI}})$$

$\mathbb{V}_{\mathrm{SI}}$ ... subdirectly irreducible algebras in $\mathbb{V}$
$\mathbb{Q}_{\mathrm{RSI}}$ ... relatively subdirectly irreducible algebras in $\mathbb{Q}$
$\mathbf{P_{SD}}$... operator for subdirect products

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$\mathbf{P_{SD}}$... operator for subdirect products

In abstract algebraic logic this amounts to

Theorem

If L is a finitary logic then

$$\mathbf{MOD}^*(\mathrm{L}) = \mathbf{P_{SD}}(\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}})$$

Preliminaries: logic

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$Fm_{\mathcal{L}} = Fm_{\mathcal{L}}(Var)$ a set of formulas

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1. $\mathcal{L}$ an algebraic signature
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$Fm_{\mathcal{L}}(X)$ a free algebra with generator X

$Fm_{\mathcal{L}} = Fm_{\mathcal{L}}(\mathit{Var})$ a set of formulas

A logic L is a relation between sets of formulas and formulas, satisfying

We write $\Gamma \vdash_{\mathrm{L}} \varphi$ instead of $\langle \Gamma, \varphi \rangle \in \mathrm{L}$

$\varphi \vdash_{\mathrm{L}} \varphi$ (reflexivity)

$\Gamma \vdash_{\mathrm{L}} \varphi \Rightarrow \Gamma, \Delta \vdash_{\mathrm{L}} \varphi$ (monotonicity)

$\Gamma \vdash_{\mathrm{L}} \Delta$ and $\Delta \vdash_{\mathrm{L}} \varphi \Rightarrow \Gamma \vdash_{\mathrm{L}} \varphi$ (cut)

$\Gamma \vdash_{\mathrm{L}} \varphi \Rightarrow \sigma\Gamma \vdash_{\mathrm{L}} \sigma\varphi$ for each substitution σ (structurality)

L is finitary if whenever $\Gamma \vdash_{\mathrm{L}} \varphi$ then $\Gamma' \vdash_{\mathrm{L}} \varphi$ for a finite $\Gamma' \subseteq \Gamma$

Preliminaries: semantics

An $\mathcal{L}$-matrix is a pair $\langle \boldsymbol{A}, F \rangle$ with $\boldsymbol{A}$ an $\mathcal{L}$-algebra and $F \subseteq A$

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Given a class of $\mathcal{L}$-matrices $\mathbb{K}$,

the corresponding semantical consequence relation is:

$$\Gamma \models_{\mathbb{K}} \varphi \quad \iff \quad \text{for every } \langle \boldsymbol{A}, F\rangle \in \mathbb{K}, \text{ and } v \in \mathrm{Hom}(\boldsymbol{Fm}_{\mathcal{L}}, \boldsymbol{A}) \text{ if } v[\Gamma] \subseteq F \text{ then } v(\varphi) \in F$$

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$\models_{\mathbb{K}}$ is a logic.

A matrix $\mathbf{A}$ is a model of L, $\mathbf{A} \in \mathbf{MOD}(\mathrm{L})$, if $\vdash_{\mathrm{L}} \subseteq \models_{\mathbf{A}}$

$$\Gamma \vdash_{\mathrm{L}} \varphi \quad \Longleftrightarrow \quad \Gamma \models_{\mathbf{MOD}(\mathrm{L})} \varphi \qquad \text{(completeness)}$$

Preliminaries: models

Let L be a logic.

$\mathbf{MOD}^*(\mathrm{L}) = \{\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L}) \ : \ \langle \boldsymbol{A}, F\rangle$ is reduced $(\Omega_{\boldsymbol{A}}(F) = \mathrm{Id}_{\boldsymbol{A}})\}$

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L is called R(F)SI-complete when

$$\Gamma \vdash_{\mathrm{L}} \varphi \quad \Longleftrightarrow \quad \Gamma \models_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{R(F)SI}}} \varphi$$

L is called (finitely) subdirectly representable when

$$\mathbf{MOD}^*(\mathrm{L}) = \mathbf{P_{SD}}(\mathbf{MOD}^*(\mathrm{L})_{\mathrm{R(F)SI}})$$

IPEP AND CIPEP

Closure systems in AAL: Let L be a logic.

1. For every algebra $\boldsymbol{A}$, the closure system of all L-filters is

$$\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A}) = \{F \subseteq A \ : \ \langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})\}$$

2. $\mathrm{Th}(\mathrm{L})$ is the closure system of all L-theories (deductively closed sets of formulas).

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Let $\mathcal{C}$ be a closure system on a set A.

- $\mathcal{B} \subseteq \mathcal{C}$ is a basis of $\mathcal{C}$ when every $X \in \mathcal{C}$ is an intersection of members from $\mathcal{B}$.
- $X \in \mathcal{C}$ is (completely) $\cap$-prime if it is (completely) $\cap$-irreducible in $\mathcal{C}$.
- $\mathcal{C}$ has the (completely) intersection-prime extension property, (C)IPEP, if the (completely) $\cap$-prime members form a basis of $\mathcal{C}$.

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L has $(\mathrm{C})\mathrm{IPEP}$ if $\mathrm{Th}(\mathrm{L})$ does.
L has the transferred $(\mathrm{C})\mathrm{IPEP}$, τ-$(\mathrm{C})\mathrm{IPEP}$, if $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ has $(\mathrm{C})\mathrm{IPEP}$ for every algebra $\boldsymbol{A}$.

Hierarchy of infinitary logics

$$\begin{tikzcd}
\text{Finitary} \arrow[r, shift left] \arrow[dr, shift left] & \tau\text{-IPEP} \arrow[l, shift left, "/" marking] \arrow[r, shift left] \arrow[d, shift left, "/" marking] & \text{IPEP} \arrow[l, shift left, "/" marking] \arrow[r, shift left] \arrow[d, shift left, "/" marking] & \text{RFSI-complete} \arrow[l, shift left, "/" marking] \arrow[d, shift left, "/" marking] \\
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\end{tikzcd}
$$

THEOREM

For every protoalgebraic logic L, *the following are equivalent:*

- L *is (finitely) subdirectly representable,* $(\mathbf{MOD}^*(\mathrm{L}) = \mathbf{P}_{\mathbf{SD}}(\mathbf{MOD}^*(\mathrm{L})_{\mathrm{R(F)SI}}))$
- L *has the* τ-(C)IPEP,
- $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{Fm}_{\boldsymbol{\mathcal{L}}}(\kappa))$ *has the* (C)IPEP *for every cardinal* κ.

Examples: product logic

Language $\{\to, \&, \overline{0}\}$ with *countable* set Var

Let $[0,1]_\Pi$ be the standard product algebra

- Universe $[0,1]$ of reals,
- $a \to^{[0,1]_\Pi} b = \min\{1, b/a\}$
- $a \,\&^{[0,1]_\Pi}\, b = a \cdot b$, and $\overline{0}^{[0,1]_\Pi} = 0$.

The infinitary product logic, Π_∞, is semantically given by the matrix $\langle [0,1]_\Pi, \{1\}\rangle$.

$$\Gamma \vdash_{\Pi_\infty} \varphi \quad \Longleftrightarrow \quad \begin{array}{l}\text{for every } v \in \mathrm{Hom}(\boldsymbol{Fm}_{\mathcal{L}}, [0,1]_\Pi) \\ \text{if } v[\Gamma] \subseteq \{1\} \text{ then } v(\varphi) = 1\end{array}$$

Examples: Łukasiewicz logic

Language $\{\to, \&, \overline{0}\}$ with *countable* set Var.

Let $[0,1]_{Ł}$ be the standard Łukasiewicz algebra:

- Universe $[0,1]$ of reals,
- $a \to^{[0,1]_{Ł}} b = \min\{1 - a + b, 1\}$
- $a \,\&^{[0,1]_{Ł}}\, b = \max\{a + b - 1, 0\}$, and $\overline{0}^{[0,1]_{Ł}} = 0$.

The infinitary Łukasiewicz logic, $Ł_\infty$, is semantically given by the matrix $\langle [0,1]_{Ł}, \{1\} \rangle$.

$$\Gamma \vdash_{Ł_\infty} \varphi \quad \iff \quad \begin{array}{l}\text{for every } v \in \mathrm{Hom}(\boldsymbol{Fm}_{\mathcal{L}}, [0,1]_{Ł}) \\ \text{if } v[\Gamma] \subseteq \{1\} \text{ then } v(\varphi) = 1\end{array}$$

Properties: infinitary product logic

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Theorem

The logic Π_∞ is not even finitely subdirectly representable (equivalently it does not have τ-IPEP). That is

$$\mathbf{MOD}^*(\Pi_\infty) \neq \mathbf{P_{SD}}(\mathbf{MOD}^*(\Pi_\infty)_{\mathrm{RFSI}}).$$

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COROLLARY

For its equivalent algebraic semantics, $\mathbf{ALG}^*(\Pi_\infty)$, we obtain

$$\mathbf{ALG}^*(\Pi_\infty) \neq \mathbf{P_{SD}}(\mathbf{ALG}^*(\Pi_\infty)_{\mathrm{R(F)SI}}).$$

Properties: infinitary Łukasiewicz logic

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Theorem

$Ł_\infty$ *has the* τ-CIPEP *and is subdirectly representable, that is*

$$\mathbf{MOD}^*(Ł_\infty) = \mathbf{P_{SD}}(\mathbf{MOD}^*(Ł_\infty)_{\mathrm{RSI}})$$

in particular, it is representable by chains.

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COROLLARY

The equivalent algebraic semantics of $Ł_\infty$, $\mathbf{ALG}^*(Ł_\infty)$, is not a quasivariety and yet

$$\mathbf{ALG}^*(Ł_\infty) = \mathbf{P_{SD}}(\mathbf{ALG}^*(Ł_\infty)_{\mathrm{RSI}}).$$

Logic of the standard MV-chain: cardinality

Lemma

The logic of $[0,1]_{Ł}$ in κ-many variables, $\vdash$, has cardinality at most $\aleph_1$.

Proof.

Assume $\Gamma \vdash \varphi$. We find a countable $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash \varphi$:

Consider the (compact) product topology $[0,1]^\kappa$ and define sets

$$\begin{aligned} \mathrm{SAT}(\varphi) &= \{v \in [0,1]^\kappa : v(\varphi) = 1\} \\ \mathrm{NSAT}(\varphi) &= \{v \in [0,1]^\kappa : v(\varphi) \neq 1\} \\ \mathrm{SAT_q}(\varphi) &= \{v \in [0,1]^\kappa : v(\varphi) > q\} \end{aligned}$$

$$\Gamma \vdash \varphi \iff \bigcup_{\gamma \in \Gamma} \mathrm{NSAT}(\gamma) \cup \mathrm{SAT}(\varphi) = [0,1]^\kappa, \qquad (*)$$

The connectives of Łukasiewicz logic are continuous, thus $\mathrm{NSAT}(\varphi)$ and $\mathrm{SAT}_q(\varphi)$ are open sets in $[0,1]^\kappa$ ($\varphi : [0,1]^\kappa \to [0,1]$ s.t. $v \mapsto v(\varphi)$).

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$$\bigcup_{\gamma\in\Gamma} \mathrm{NSAT}(\gamma) \cup \mathrm{SAT}(\varphi) = [0,1]^{\kappa}, \tag{1}$$

since $\mathrm{SAT}(\varphi) \subseteq \mathrm{SAT}_q(\varphi)$, we get

$$\bigcup_{\gamma\in\Gamma} \mathrm{NSAT}(\gamma) \cup \mathrm{SAT}_q(\varphi) = [0,1]^{\kappa}, \tag{2}$$

Let $\Gamma_q \subseteq \Gamma$ generate finite subcover of (2) and set $\Gamma' = \bigcup_{q\in(0,1)} \Gamma_q$.

$$\{1\} = \bigcap_{q\in(0,1)} \uparrow q \quad \text{implies} \quad \bigcup_{\gamma\in\Gamma'} \mathrm{NSAT}(\gamma) \cup \mathrm{SAT}(\varphi) = [0,1]^{\kappa},$$

consequently, by (*), $\Gamma' \vdash \varphi$. □

Logic with τ-IPEP which is not RSI-complete

$\mathcal{L} = \{\rightarrow, \&, \overline{0}\} \cup \{\bar{q} : q \in (0,1] \cap \mathbb{Q}\}$ and $Var = \omega$.

$[0,1]_{Ł}^{\mathbb{Q}}$ is $[0,1]_{Ł}$ with naturally defined rational constants.

Let L be the logic preserving degrees of truth in $[0,1]_{Ł}^{\mathbb{Q}}$, i.e.

$$\Gamma \vdash_{\mathrm{L}} \varphi \quad \Longleftrightarrow \quad \bigwedge v[\Gamma] \leq v(\varphi), \text{ for all } v \in \mathrm{Hom}(\boldsymbol{Fm}_{\mathcal{L}}, [0,1]_{Ł}^{\mathbb{Q}}).$$

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- L is equivalential (with implication $x \Rightarrow y = \{(x \to y)^n : n \in \omega\}$), but not algebraizable.
- L has the τ-IPEP (and is finitely subdirectly representable).
- L has no RSI-models, thus it is not RSI-complete.

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Thank you,

Enjoy Prague!

Cardinality upper bound

Theorem

Suppose λ is a regular cardinal and $\mathbb{K}$ is a class of matrices, such that $|\mathbb{K}| < \lambda$. Further suppose that for every $\langle \boldsymbol{A}, F \rangle \in \mathbb{K}$:

1. *There is a compact topology τ on A such that all of the connectives are continuous w.r.t. τ,*
2. *F can be written as an intersection of strictly less λ open sets in τ,*
3. *$A \setminus F \in \tau$,*

then $\mathrm{L}_{\mathbb{K},\kappa}$ has cardinality at most λ for every infinite κ.