Intermediate logics admitting structural hypersequent calculi

Frederik Möllerström Lauridsen

University of Amsterdam (ILLC)

TACL
June 28, 2017
1. Why is it that some logics are difficult—or impossible—to capture with “good” and “simple” proof calculi?

2. Are there maybe some semantic obstructions for obtaining such “good” and “simple” proof calculi?
Proof theory for non-classical logics

Questions

1. Why is it that some logics are difficult—or impossible—to capture with “good” and “simple” proof calculi?
Proof theory for non-classical logics

Questions

1. Why is it that some logics are difficult—or impossible—to capture with “good” and “simple” proof calculi?
2. Are there maybe some semantic obstructions for obtaining such “good” and “simple” proof calculi?
Intermediate logics

We obtain intuitionistic propositional logic \( \text{Int} \) by dropping the law of excluded middle \( p \lor \neg p \) from classical propositional logic \( \text{Cl} \).

Propositional logics \( L \) such that \( \text{Int} \subseteq L \subseteq \text{Cl} \) are called intermediate logics.

Semantics

For every intermediate logic \( L \), there is a variety \( V(L) \) of Heyting algebras such that \( \phi \models_L \iff V(L) \models_\phi \), and conversely.
Intermediate logics

We obtain *intuitionistic propositional logic* $\text{Int}$ by dropping the law of excluded middle $p \lor \neg p$ from *classical propositional logic* $\text{Cl}$.
Intermediate logics

We obtain *intuitionistic propositional logic* $\text{Int}$ by dropping the law of excluded middle $p \lor \neg p$ from *classical propositional logic* $\text{Cl}$.

Propositional logics $L$ such that

$$\text{Int} \subseteq L \subseteq \text{Cl}$$

are called *intermediate logics*. 
Intermediate logics

We obtain *intuitionistic propositional logic* $\text{Int}$ by dropping the law of excluded middle $p \lor \neg p$ from *classical propositional logic* $\text{Cl}$.

Propositional logics $L$ such that

$$\text{Int} \subseteq L \subseteq \text{Cl}$$

are called *intermediate logics*.

Semantics
Intermediate logics

We obtain *intuitionistic propositional logic* \( \text{Int} \) by dropping the law of excluded middle \( p \lor \neg p \) from *classical propositional logic* \( \text{Cl} \).

Propositional logics \( L \) such that

\[
\text{Int} \subseteq L \subseteq \text{Cl}
\]

are called *intermediate logics*.

**Semantics**

For every intermediate logic \( L \) there is a variety \( \mathcal{V}(L) \) of Heyting algebras such that \( \varphi \in L \) iff \( \mathcal{V}(L) \models \varphi \approx 1 \), and conversely.
Gentzen’s sequent calculus $\text{LJ}$ for $\text{Int}$
Gentzen’s sequent calculus LJ for \textbf{Int}

\[
\varphi \in \text{Int} \text{ iff } \vdash_{LJ} \Rightarrow \varphi \text{ iff } \vdash_{LJ}^{\text{cut free}} \Rightarrow \varphi.
\]
Gentzen’s sequent calculus \( \text{LJ} \) for \( \text{Int} \)

\[ \varphi \in \text{Int} \iff \vdash_{\text{LJ}} \varphi \iff \vdash_{\text{cutfree}} \varphi. \]

**Sequent calculi for intermediate logic**

Adding additional axioms or rules to \( \text{LJ} \) usually breaks the cut-elimination procedure.
Gentzen’s sequent calculus LJ for Int

\[ \varphi \in \text{Int} \iff \vdash_{LJ} \Rightarrow \varphi \iff \vdash_{LJ}^{\text{cutfree}} \Rightarrow \varphi. \]

Sequent calculi for intermediate logic

Adding additional axioms or rules to LJ usually breaks the cut-elimination procedure. With some exceptions:
Gentzen’s sequent calculus LJ for \( \text{Int} \)

\[
\varphi \in \text{Int} \quad \text{iff} \quad \vdash_{\text{LJ}} \Rightarrow \varphi \quad \text{iff} \quad \vdash_{\text{LJ}}^{\text{cutfree}} \Rightarrow \varphi.
\]

**Sequent calculi for intermediate logic**

Adding additional axioms or rules to \( \text{LJ} \) usually breaks the cut-elimination procedure. With some exceptions:

1. Cut-free sequent calculi for \( \text{LC} \); (Sonobe 1975, Corsi 1989);
Gentzen’s sequent calculus LJ for Int

\[ \varphi \in \text{Int} \iff \vdash \text{LJ} \Rightarrow \varphi \iff \vdash_{\text{cut-free}} \text{LJ} \Rightarrow \varphi. \]

Sequent calculi for intermediate logic

Adding additional axioms or rules to \( \text{LJ} \) usually breaks the cut-elimination procedure. With some exceptions:

1. Cut-free sequent calculi for \( \text{LC} \); (Sonobe 1975, Corsi 1989);

2. Cut-free sequent calculi for \( \text{LC}, \text{KC}, \text{LC}_2, \text{BD}_2, \text{Sm} \); (Avellone et al. 1999).
Gentzen’s sequent calculus $\text{LJ}$ for $\text{Int}$

\[ \varphi \in \text{Int} \iff \vdash_{\text{LJ}} \varphi \iff \vdash_{\text{LJ}}^{\text{cut-free}} \varphi. \]

**Sequent calculi for intermediate logic**

Adding additional axioms or rules to $\text{LJ}$ usually breaks the cut-elimination procedure. With some exceptions:

1. Cut-free sequent calculi for $\text{LC}$; (Sonobe 1975, Corsi 1989);

2. Cut-free sequent calculi for $\text{LC}$, $\text{KC}$, $\text{LC}_2$, $\text{BD}_2$, $\text{Sm}$; (Avellone et al. 1999).

**Negative results**
Gentzen’s sequent calculus LJ for Int

\[ \varphi \in \text{Int} \iff \vdash_{LJ} \Rightarrow \varphi \iff \vdash_{LJ}^{\text{cutfree}} \Rightarrow \varphi. \]

Sequent calculi for intermediate logic

Adding additional axioms or rules to \( \text{LJ} \) usually breaks the cut-elimination procedure. With some exceptions:

1. Cut-free sequent calculi for \( \text{LC} \); (Sonobe 1975, Corsi 1989);
2. Cut-free sequent calculi for \( \text{LC}, \text{KC}, \text{LC}_2, \text{BD}_2, \text{Sm} \); (Avellone et al. 1999).

Negative results

1. No proper intermediate logic admits a \textit{structural} extension of \( \text{LJ} \) (Ciabattoni et al. 2008);
Gentzen’s sequent calculus $\text{LJ}$ for $\text{Int}$

$$\varphi \in \text{Int} \iff \vdash_{\text{LJ}} \Rightarrow \varphi \iff \vdash_{\text{LJ}}^{\text{cutfree}} \Rightarrow \varphi.$$ 

Sequent calculi for intermediate logic

Adding additional axioms or rules to $\text{LJ}$ usually breaks the cut-elimination procedure. With some exceptions:

1. Cut-free sequent calculi for $\text{LC}$; (Sonobe 1975, Corsi 1989);
2. Cut-free sequent calculi for $\text{LC}$, $\text{KC}$, $\text{LC}_2$, $\text{BD}_2$, $\text{Sm}$; (Avellone et al. 1999).

Negative results

1. No proper intermediate logic admits a *structural* extension of $\text{LJ}$ (Ciabattoni et al. 2008);
2. Few intermediate logics with *focused* terminating sequent calculi (Iemhoff 2017).
Hypersequent calculi
Hypersequent calculi

Definition (Mints 1968, Pottinger 1983, Avron 1987)

\[ \Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Gamma_n \Rightarrow \Pi_n \]
Hypersequent calculi

**Definition (Mints 1968, Pottinger 1983, Avron 1987)**

\[ \Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Gamma_n \Rightarrow \Pi_n \]

We have a hypersequent calculus HLJ for Int.

\[
\begin{array}{c}
H \mid s_1 \hspace{2cm} \ldots \hspace{2cm} \ldots \hspace{2cm} H \mid s_n \\
\hline
H \mid s_0
\end{array}
\]

\( (hr) \)

\[
\begin{array}{c}
s_1 \hspace{2cm} \ldots \hspace{2cm} s_n \hspace{2cm} (r) \\
\hline
s_0
\end{array}
\]
Definition (Mints 1968, Pottinger 1983, Avron 1987)

\[ \Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Gamma_n \Rightarrow \Pi_n \]

We have a hypersequent calculus HLJ for Int.

\[
\frac{s_1 \ldots s_n}{s_0} \quad (r) \quad \sim \quad \frac{H \mid s_1 \ldots s_n}{H \mid s_0} \quad (hr)
\]

\[
\frac{H \mid \Gamma \Rightarrow \Pi \mid \Gamma \Rightarrow \Pi}{H \mid \Gamma \Rightarrow \Pi} \quad (EC)
\]

\[
\frac{H \mid \Gamma \Rightarrow \Pi}{H \mid \Gamma \Rightarrow \Pi} \quad (EW)
\]
Analytic hypersequent calculi for LC and KC
Analytic hypersequent calculi for LC and KC

\[ \text{LC} = \text{Int} \ + \ (p \to q) \lor (q \to p) \quad \text{KC} = \text{Int} \ + \ \neg p \lor \neg \neg p. \]
Analytic hypersequent calculi for LC and KC

LC = Int + (p → q) ∨ (q → p)  
KC = Int + ¬p ∨ ¬¬p.

Examples

\[
\frac{H \mid \Gamma_1, \Sigma_2 \Rightarrow \Pi_1 \quad H \mid \Gamma_2, \Sigma_1 \Rightarrow \Pi_2}{H \mid \Gamma_1, \Sigma_1 \Rightarrow \Pi_1 \mid \Gamma_2, \Sigma_2 \Rightarrow \Pi_2} \quad (lc)
\]

\[
\frac{H \mid \Gamma, \Sigma \Rightarrow}{H \mid \Gamma \Rightarrow \mid \Sigma \Rightarrow} \quad (lq)
\]
The substructural hierarchy

We may define a hierarchy of formulas as follows:

\[ P_0 = N_0 = \text{Prop}, \]

and

\[ P_{n+1} ::= \top_j ?_j N_n ?_j P_{n+1} ^ P_{n+1} j P_{n+1} _ P_{n+1} \]

\[ N_{n+1} ::= ?_j \top_j P_n ?_j N_n ^ N_n j P_{n+1} ! N_{n+1} \]

Remark

1. Over \( \text{Int} \) the “hierarchy” collapses above the level \( N_3 \);
2. For every formula \( \phi \) in \( N_2 \) we have that \( \text{Int} \phi \leq f \text{Form} \); \( \text{Int} g \).
The substructural hierarchy

We may define a hierarchy of formulas as follows:
The substructural hierarchy

We may define a hierarchy of formulas as follows: \( \mathcal{P}_0 = \mathcal{N}_0 = \text{Prop} \), and

\[
\begin{align*}
\mathcal{P}_{n+1} &::= \top \mid \bot \mid \mathcal{N}_n \mid \mathcal{P}_{n+1} \land \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \lor \mathcal{P}_{n+1} \\
\mathcal{N}_{n+1} &::= \bot \mid \top \mid \mathcal{P}_n \mid \mathcal{N}_{n+1} \land \mathcal{N}_{n+1} \mid \mathcal{P}_{n+1} \rightarrow \mathcal{N}_{n+1}
\end{align*}
\]
The substructural hierarchy

We may define a hierarchy of formulas as follows: $\mathcal{P}_0 = \mathcal{N}_0 = \text{Prop}$, and

\[
\begin{align*}
\mathcal{P}_{n+1} & ::= \top | \bot | \mathcal{N}_n | \mathcal{P}_{n+1} \land \mathcal{P}_{n+1} | \mathcal{P}_{n+1} \lor \mathcal{P}_{n+1} \\
\mathcal{N}_{n+1} & ::= \bot | \top | \mathcal{P}_n | \mathcal{N}_{n+1} \land \mathcal{N}_{n+1} | \mathcal{P}_{n+1} \rightarrow \mathcal{N}_{n+1}
\end{align*}
\]

**Remark**

1. Over $\text{Int}$ the “hierarchy” collapses above the level $\mathcal{N}_3$.
The substructural hierarchy

We may define a hierarchy of formulas as follows: $\mathcal{P}_0 = \mathcal{N}_0 = \text{Prop}$, and

$$
\mathcal{P}_{n+1} ::= \top | \bot | \mathcal{N}_n | \mathcal{P}_{n+1} \wedge \mathcal{P}_{n+1} | \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1}
$$

$$
\mathcal{N}_{n+1} ::= \bot | \top | \mathcal{P}_n | \mathcal{N}_{n+1} \wedge \mathcal{N}_{n+1} | \mathcal{P}_{n+1} \rightarrow \mathcal{N}_{n+1}
$$

Remark

1. Over $\text{Int}$ the “hierarchy” collapses above the level $\mathcal{N}_3$;
2. For every formula $\varphi \in \mathcal{N}_2$ we have that $\text{Int} + \varphi \in \{\text{Form, Int}\}$. 
Theorem (Ciabattoni et al. 2008, 2017)

There is an effective procedure transforming any $P_3$-axiom $\varphi$ into a finite set of “equivalent” structural hypersequent rule $\mathcal{R}$ such that cut-admissibility is preserved when adding $\mathcal{R}$ to HLJ.

Here “equivalent” means equivalent on subdirectly irreducible Heyting algebras. So $HLJ + \mathcal{R}$ will be a calculus for $Int + \varphi$.

Observation
At least countably many proper intermediate logics are axiomatisable by $P_3$-formulas. E.g., $BW_n$; $BTW_n$; $BC_n$, for $n \in \mathbb{N}$. 

Systematic proof theory
Theorem (Ciabattoni et al. 2008, 2017)

There is an effective procedure transforming any $P_3$-axiom $\varphi$ into a finite set of “equivalent” structural hypersequent rule $R$ such that cut-admissibility is preserved when adding $R$ to HLJ.

Here “equivalent” means equivalent on subdirectly irreducible Heyting algebras.
Theorem (Ciabattoni et al. 2008, 2017)

There is an effective procedure transforming any $P_3$-axiom $\varphi$ into a finite set of “equivalent” structural hypersequent rule $R$ such that cut-admissibility is preserved when adding $R$ to $HLJ$.

Here “equivalent” means equivalent on subdirectly irreducible Heyting algebras. So $HLJ + R$ will be a calculus for $Int + \varphi$. 
Systematic proof theory

Theorem (Ciabattoni et al. 2008, 2017)

There is an effective procedure transforming any $P_3$-axiom $\varphi$ into a finite set of “equivalent” structural hypersequent rule $R$ such that cut-admissibility is preserved when adding $R$ to HLJ.

Here “equivalent” means equivalent on subdirectly irreducible Heyting algebras. So $HLJ + R$ will be a calculus for $Int + \varphi$.

Observation

At least countably many proper intermediate logics are axiomatisable by $P_3$-formulas.
Theorem (Ciabattoni et al. 2008, 2017)

There is an effective procedure transforming any $\mathcal{P}_3$-axiom $\varphi$ into a finite set of “equivalent” structural hypersequent rule $\mathcal{R}$ such that cut-admissibility is preserved when adding $\mathcal{R}$ to HLJ.

Here “equivalent” means equivalent on subdirectly irreducible Heyting algebras. So $\text{HLJ} + \mathcal{R}$ will be a calculus for $\text{Int} + \varphi$.

Observation

At least countably many proper intermediate logics are axiomatisable by $\mathcal{P}_3$-formulas. E.g., $\text{BW}_n$, $\text{BTW}_n$, $\text{BC}_n$, for $n \in \mathbb{N}$. 

A problem with syntactic classifications

Given an intermediate logic $L := \text{Int} + \varphi$ with $\varphi \not\in \mathcal{P}_3$ there might exist $\psi \in \mathcal{P}_3$ such that $L = \text{Int} + \psi$. 
A problem with syntactic classifications

Given an intermediate logic $L := \text{Int} + \varphi$ with $\varphi \not\in \mathcal{P}_3$ there might exist $\psi \in \mathcal{P}_3$ such that $L = \text{Int} + \psi$. For example:

\[
\text{BTW}_n = \text{Int} + \bigwedge_{0 \leq i < j \leq n} \left( \neg (\neg p_i \land \neg p_j) \rightarrow \bigvee_{i=0}^{n} (\neg p_i \rightarrow \bigvee_{j \neq i} \neg p_j) \right)
\]

\[
= \text{Int} + \bigvee_{i=0}^{n} \left( \bigwedge_{j < i} p_j \rightarrow \neg \neg p_i \right).
\]
A problem with syntactic classifications

Given an intermediate logic $L := \text{Int} + \varphi$ with $\varphi \not\in \mathcal{P}_3$ there might exist $\psi \in \mathcal{P}_3$ such that $L = \text{Int} + \psi$. For example:

$$
\text{BTW}_n = \text{Int} + \bigwedge_{0 \leq i < j \leq n} \left( \neg(\neg p_i \land \neg p_j) \rightarrow \bigvee_{i=0}^{n} (\neg p_i \rightarrow \bigvee_{j \neq i} \neg p_j) \right)
$$

$$
= \text{Int} + \bigvee_{i=0}^{n} \left( \bigwedge_{j < i} p_j \rightarrow \neg \neg p_i \right).
$$

We need intrinsic semantic characterisations of logics with an $\mathcal{P}_3$-axiomatisation.
Theorem (Ciabattoni et al. 2008)

There is an effective procedure transforming any structural hypersequent rule \((r)\) into an equivalent structural hypersequent rule \((r')\) such that cut-admissibility is preserved by adding \((r')\) to HLJ.

Theorem (Ciabattoni et al. 2008/2017)

Let \(L\) be an intermediate logic. Then the following are equivalent:

1. \(L\) is axiomatised by \(P_3\)-formulas;
2. \(L\) admits an analytic hypersequent calculus extending HLJ with structural rules;
3. \(L\) admits a hypersequent calculus extending HLJ with structural rules.

Thus we only need to consider intermediate logics with a structural hypersequent calculus.

Rule completion
Theorem (Ciabattoni et al. 2008)

There is an effective procedure transforming any structural hypersequent rule \((r)\) into an equivalent structural hypersequent rule \((r')\) such that cut-admissibility is preserved by adding \((r')\) to \(\text{HLJ}\).
Rule completion

Theorem (Ciabattoni et al. 2008)

There is an effective procedure transforming any structural hypersequent rule \((r)\) into an equivalent structural hypersequent rule \((r')\) such that cut-admissibility is preserved by adding \((r')\) to \(\text{HLJ}\).

Theorem (Ciabattoni et al. 2008/2017)

Let \(L\) be an intermediate logic. Then the following are equivalent:

1. \(L\) is axiomatisable by \(P_3\)-formulas;
Rule completion

Theorem (Ciabattoni et al. 2008)

There is an effective procedure transforming any structural hypersequent rule \((r)\) into an equivalent structural hypersequent rule \((r')\) such that cut-admissibility is preserved by adding \((r')\) to HLJ.

Theorem (Ciabattoni et al. 2008/2017)

Let \(L\) be an intermediate logic. Then the following are equivalent:

1. \(L\) is axiomatisable by \(P_3\)-formulas;

2. \(L\) admits an analytic hypersequent calculus extending HLJ with structural rules;
Rule completion

Theorem (Ciabattoni et al. 2008)

There is an effective procedure transforming any structural hypersequent rule \((r)\) into an equivalent structural hypersequent rule \((r')\) such that cut-admissibility is preserved by adding \((r')\) to HLJ.

Theorem (Ciabattoni et al. 2008/2017)

Let \(L\) be an intermediate logic. Then the following are equivalent:

1. \(L\) is axiomatisable by \(P_3\)-formulas;
2. \(L\) admits an analytic hypersequent calculus extending HLJ with structural rules;
3. \(L\) admits a hypersequent calculus extending HLJ with structural rules.
Rule completion

Theorem (Ciabattoni et al. 2008)

There is an effective procedure transforming any structural hypersequent rule \((r)\) into an equivalent structural hypersequent rule \((r')\) such that cut-admissibility is preserved by adding \((r')\) to HLJ.

Theorem (Ciabattoni et al. 2008/2017)

Let \(L\) be an intermediate logic. Then the following are equivalent:

1. \(L\) is axiomatisable by \(P_3\)-formulas;

2. \(L\) admits an analytic hypersequent calculus extending HLJ with structural rules;

3. \(L\) admits a hypersequent calculus extending HLJ with structural rules.

Thus we only need to consider intermediate logics with a structural hypersequent calculus.
Structural hypersequent rules and universal clauses

Observation (Ciabattoni et al. 2017)

We have a correspondence between structural hypersequent rules and universal clauses in the $(0; ^; 1)$-reduct of the language of Heyting algebras.

Examples

$H_j (1 H_j (2 H_j (1 l c) (2 H_j (1 j (2 H_j (1 x x' 2 y 1) x 2 x' 2 y 2) = x 1 x' 1 y 1$ or $x 2 x' 2 y 2) = x 1 x' 1 y 1$ or $x 2 x' 2 y 2) = x 1 x' 1 y 1$.

Having
Structural hypersequent rules and universal clauses

Observation (Ciabattoni et al. 2017)

We have a correspondence between structural hypersequent rules and universal clauses in the \((0, \land, 1)\)-reduct of the language of Heyting algebras.
Structural hypersequent rules and universal clauses

Observation (Ciabattoni et al. 2017)

We have a correspondence between structural hypersequent rules and
universal clauses in the $\langle 0, \land, 1 \rangle$-reduct of the language of Heyting algebras.

Examples

\[
\frac{H \mid \Gamma_1, \Sigma_2 \Rightarrow \Pi_1 \quad H \mid \Gamma_2, \Sigma_1 \Rightarrow \Pi_2}{H \mid \Gamma_1, \Sigma_1 \Rightarrow \Pi_1 \mid \Gamma_2, \Sigma_2 \Rightarrow \Pi_2} \quad (lc)
\]

\[
x_1 \land x'_2 \leq y_1 \text{ and } x_2 \land x'_1 \leq y_2 \implies x_1 \land x'_1 \leq y_1 \text{ or } x_2 \land x'_2 \leq y_2.
\]
Observation (Ciabattoni et al. 2017)

We have a correspondence between structural hypersequent rules and universal clauses in the \((0, \wedge, 1)\)-reduct of the language of Heyting algebras.

Examples

\[
\begin{align*}
\frac{H \mid \Gamma_1, \Sigma_2 \Rightarrow \Pi_1 \quad H \mid \Gamma_2, \Sigma_1 \Rightarrow \Pi_2}{H \mid \Gamma_1, \Sigma_1 \Rightarrow \Pi_1 \mid \Gamma_2, \Sigma_2 \Rightarrow \Pi_2} \quad (lc) \\
x_1 \wedge x_2' \leq y_1 \text{ and } x_2 \wedge x_1' \leq y_2 \implies x_1 \wedge x_1' \leq y_1 \text{ or } x_2 \wedge x_2' \leq y_2.
\end{align*}
\]

\[
\frac{H \mid \Gamma, \Sigma \Rightarrow}{H \mid \Gamma \Rightarrow \mid \Sigma \Rightarrow} \quad (lq)
\]

\[
x_1 \wedge x_2 \leq 0 \implies x_1 \leq 0 \text{ or } x_2 \leq 0.
\]
Observation
Suppose that $L$ admits a structural hypersequent calculus. Then, for Heyting algebras $A$, $B$, if $B \models \mu(L)$ and $A \not\models 0;^1;1^1 B$, then $A \models \mu(L)$.

Definition
An intermediate logic satisfying $\mu$ is said to be $(0;^1;1^1)$-stable.

Lemma
If $L$ is $(0;^1;1^1)$-stable then $\mu(L)$ is generated by a universal class of Heyting algebras axiomatised by a collection of universal $(0;^1;1^1)$-clauses. The proof uses "canonical" clauses $q_{0;^1;1^1}(A)$ associated with finite Heyting algebras $B_j = q_{0;^1;1^1}(A)$.
(0, ∨, 1)-stable logics

Observation

Suppose that $L$ admits a structural hypersequent calculus. Then, for Heyting algebras $A, B$,

$$\text{If } B \in \mathbb{V}(L)_{si} \text{ and } A \rightarrow_{0, ∨, 1} B \text{ then } A \in \mathbb{V}(L).$$

(†)
Observation

Suppose that $L$ admits a structural hypersequent calculus. Then, for Heyting algebras $A, B$,

$$\text{If } B \in \mathcal{V}(L)_{si} \text{ and } A \not\rightarrow_{0, \land, 1} B \text{ then } A \in \mathcal{V}(L). \quad (\dagger)$$

Definition

An intermediate logic satisfying $(\dagger)$ is said to be $(0, \land, 1)$-stable.
(0, ∧, 1)-stable logics

Observation

Suppose that $L$ admits a structural hypersequent calculus. Then, for Heyting algebras $A, B$,

$$\text{If } B \in \mathcal{V}(L)_{si} \text{ and } A \leftrightarrow_{0,\wedge,1} B \text{ then } A \in \mathcal{V}(L). \quad (\dagger)$$

Definition

An intermediate logic satisfying $(\dagger)$ is said to be $(0, \wedge, 1)$-stable.

Lemma

If $L$ is $(0, \wedge, 1)$-stable then $\mathcal{V}(L)$ is generated by a universal class of Heyting algebras axiomatised by a collection of universal $(0, \wedge, 1)$-clauses.
(0, ∨, 1)-stable logics

**Observation**

Suppose that $L$ admits a structural hypersequent calculus. Then, for Heyting algebras $A, B$,

\[
\text{If } B \in \mathbb{V}(L)_{si} \text{ and } A \rightarrow_{0,\forall,1} B \text{ then } A \in \mathbb{V}(L). \quad (\dagger)
\]

**Definition**

An intermediate logic satisfying (\dagger) is said to be (0, ∨, 1)-stable.

**Lemma**

If $L$ is (0, ∨, 1)-stable then $\mathbb{V}(L)$ is generated by a universal class of Heyting algebras axiomatised by a collection of universal (0, ∨, 1)-clauses.

The proof uses “canonical” clauses $q_{0,\forall,1}(A)$ associated with finite Heyting algebras.
(0, ∧, 1)-stable logics

Observation

Suppose that \( L \) admits a structural hypersequent calculus. Then, for Heyting algebras \( A, B \),

\[
\text{If } B \in \mathcal{V}(L)_{si} \text{ and } A \rightarrow_{0,\wedge,1} B \text{ then } A \in \mathcal{V}(L). \tag{\dagger}
\]

Definition

An intermediate logic satisfying (\dagger) is said to be \((0, \wedge, 1)\)-stable.

Lemma

If \( L \) is \((0, \wedge, 1)\)-stable then \( \mathcal{V}(L) \) is generated by a universal class of Heyting algebras axiomatised by a collection of universal \((0, \wedge, 1)\)-clauses.

The proof uses “canonical” clauses \( q_{0,\wedge,1}(A) \) associated with finite Heyting algebras.

\[
B \models q_{0,\wedge,1}(A) \iff A \notightarrow_{0,\wedge,1} B.
\]
Theorem

Let $L$ be an intermediate logic. The following are equivalent:

1. $L$ is $P_3$-axiomatisable;
2. $L$ has an analytic structural hypersequent calculus extending $HLJ$;
3. $L$ is $(0; \wedge; 1)$-stable.

Corollary

None of the logics $BD_n$, for $n \geq 2$, can be captured by a structural extension of $HLJ$. 
Theorem

Let $L$ be an intermediate logic. The following are equivalent:
Theorem

Let $L$ be an intermediate logic. The following are equivalent:

1. $L$ is $P_3$-axiomatisable;
Theorem

Let $L$ be an intermediate logic. The following are equivalent:

1. $L$ is $\mathcal{P}_3$-axiomatisable;
2. $L$ has an analytic structural hypersequent calculus extending $\text{HLJ}$;
Theorem

Let $L$ be an intermediate logic. The following are equivalent:

1. $L$ is $P_3$-axiomatisable;
2. $L$ has an analytic structural hypersequent calculus extending $\text{HLJ}$;
3. $L$ is $(0, \wedge, 1)$-stable.

Corollary

None of the logics $\text{BD}_n$, for $n \geq 2$, can be captured by a structural extension of $\text{HLJ}$. 

13
Theorem

Let $L$ be an intermediate logic. The following are equivalent:

1. $L$ is $P_3$-axiomatisable;
2. $L$ has an analytic structural hypersequent calculus extending $\text{HLJ}$;
3. $L$ is $(0, \land, 1)$-stable.

Corollary

Non of the logics $\text{BD}_n$, for $n \geq 2$, can be captured by a structural extension of $\text{HLJ}$. 
Theorem

Let $L$ be an intermediate logic. The following are equivalent:

1. $L$ is $P_3$-axiomatisable;
2. $L$ has an analytic structural hypersequent calculus extending $HLJ$;
3. $L$ is $(0;^1)$-stable;
4. $L$ is sound and complete with respect to a first-order definable class of intuitionistic Kripke frames determined by formulas of the form:

$\forall \vec{w}, \forall v \text{OR} i \forall 2^I \text{AND} j \forall 2^J \phi_{ij} (\vec{w};v)$;

where $\phi_{ij}(\vec{w};v)$ is either $wRv$ or $w = v$ for some $w \in \vec{w}$.

Compare this with the simple formulas from (Lahav 2013).
Theorem

Let $L$ be an intermediate logic. The following are equivalent:

1. $L$ is $P_3$-axiomatisable;
2. $L$ has an analytic structural hypersequent calculus extending $\text{HLJ}$;
3. $L$ is $(0, \wedge, 1)$-stable;
4. $L$ is sound and complete with respect to a first-order definable class of intuitionistic Kripke frames determined by formulas of the form:

   $\forall \vec{w} \exists v \text{ OR } i \leq j \leq I \text{ AND } i \phi_{ij} (\vec{w};v)$;

where $\phi_{ij}(\vec{w};v)$ is either $wRv$ or $w = v$ for some $w \in \vec{w}$.

Compare this with the simple formulas from (Lahav 2013).
Theorem

Let $L$ be an intermediate logic. The following are equivalent:

1. $L$ is $P_3$-axiomatisable;
2. $L$ has an analytic structural hypersequent calculus extending HLJ;
3. $L$ is $(0, \land, 1)$-stable;
4. $L$ is sound and complete with respect to a first-order definable class of intuitionistic Kripke frames determined by formulas of the form:

$\exists \vec{w} \forall v \text{ OR } i \exists j \phi_{ij} (\vec{w}; v)$;

where $\phi_{ij} (\vec{w}; v)$ is either $wRv$ or $w = v$ for some $w \in \vec{w}$.
“Semantic” characterisation II

Theorem

Let $L$ be an intermediate logic. The following are equivalent:

1. $L$ is $\mathcal{P}_3$-axiomatisable;
2. $L$ has an analytic structural hypersequent calculus extending $\text{HLJ}$;
3. $L$ is $(0, \land, 1)$-stable;
4. $L$ is sound and complete with respect to a first-order definable class of intuitionistic Kripke frames determined by formulas of the form:

$$\forall \bar{w} \exists v \text{OR}_{i \in I} \text{AND}_{j \in J_i} \varphi_{ij}(\bar{w}, v),$$
Theorem

Let $L$ be an intermediate logic. The following are equivalent:

1. $L$ is $P_3$-axiomatisable;
2. $L$ has an analytic structural hypersequent calculus extending $\text{HLJ}$;
3. $L$ is $(0, \land, 1)$-stable;
4. $L$ is sound and complete with respect to a first-order definable class of intuitionistic Kripke frames determined by formulas of the form:

$$\forall \bar{w} \exists v \text{OR}_{i \in I} \text{AND}_{j \in J} \varphi_{ij}(\bar{w}, v),$$

where $\varphi_{ij}(\bar{w}, v)$ is either $w R v$ or $w = v$ for some $w \in \bar{w}$. 

Compare this with the simple formulas from (Lahav 2013).
“Semantic” characterisation II

**Theorem**

Let $L$ be an intermediate logic. The following are equivalent:

1. $L$ is $P_3$-axiomatisable;
2. $L$ has an analytic structural hypersequent calculus extending $\text{HLJ}$;
3. $L$ is $(0, \land, 1)$-stable;
4. $L$ is sound and complete with respect to a first-order definable class of intuitionistic Kripke frames determined by formulas of the form:

$$\forall \vec{w} \exists v \text{OR}_{i \in I} \text{AND}_{j \in J_i} \varphi_{ij}(\vec{w}, v),$$

where $\varphi_{ij}(\vec{w}, v)$ is either $wRv$ or $w = v$ for some $w \in \vec{w}$.

Compare this with the simple formulas from (Lahav 2013).
Some corollaries

Let $L$ be an intermediate logic with a structural hypersequent calculus extending $\text{HLJ}$. Then

1. $L$ enjoys the finite model property;
2. $L$ is a cofinal subframe logic;
3. $L$ is Kripke complete;
4. The class of $L$-frames is an elementary class;
5. $L$ is canonical;
6. $L$ is axiomatised by $(\rightarrow, \wedge, \bot)$-formulas;
7. The class of well-connected $\forall(L)$ algebras is closed under MacNeille completion;
Some open problems

1. Is being \((0 \wedge 1)\)-stable a decidable property of intermediate logics?

2. Can we do something similar for substructural and modal logics?

3. Are similar semantic characterisations available for other proof-theoretic formalisms?
Some open problems

1. Is being $(0, \land, 1)$-stable a decidable property of intermediate logics?
Some open problems

1. Is being \((0, \land, 1)\)-stable a decidable property of intermediate logics?
2. Can we do something similar for substructural and modal logics?
Some open problems

1. Is being \((0, \land, 1)\)-stable a decidable property of intermediate logics?
2. Can we do something similar for substructural and modal logics?
3. Are similar semantic characterisations available for other proof-theoretic formalisms?
Thank you very much for your time and attention.