Intermediate logics admitting structural hypersequent calculi

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Proof theory for non-classical logics

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1. Why is it that some logics are difficult—or impossible—to capture with “good” and “simple” proof calculi?
2. Are there maybe some semantic obstructions for obtaining such “good” and “simple” proof calculi?

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Semantics

For every intermediate logic L there is a variety $\mathbb{V}(L)$ of Heyting algebras such that $\varphi \in L$ iff $\mathbb{V}(L) \models \varphi \approx 1$, and conversely.

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Negative results

1. No proper intermediate logic admits a *structural* extension of **LJ** (Ciabattoni et al. 2008);
2. Few intermediate logics with *focussed* terminating sequent calculi (Iemhoff 2017).

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$$\frac{s_1 \quad \ldots \quad s_n}{s_0}\,(r) \quad \rightsquigarrow \quad \frac{H \mid s_1 \quad \ldots \quad H \mid s_n}{H \mid s_0}\,(hr)$$

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$$\frac{H \mid \Gamma \Rightarrow \Pi \mid \Gamma \Rightarrow \Pi}{H \mid \Gamma \Rightarrow \Pi}\,(EC) \qquad \frac{H}{H \mid \Gamma \Rightarrow \Pi}\,(EW)$$

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Examples

$$\frac{H \mid \Gamma_1, \Sigma_2 \Rightarrow \Pi_1 \qquad H \mid \Gamma_2, \Sigma_1 \Rightarrow \Pi_2}{H \mid \Gamma_1, \Sigma_1 \Rightarrow \Pi_1 \mid \Gamma_2, \Sigma_2 \Rightarrow \Pi_2}\ (lc)$$

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\mathcal{P}_{n+1} &::= \top \mid \bot \mid \mathcal{N}_n \mid \mathcal{P}_{n+1} \wedge \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1} \\
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Remark

1. Over **Int** the "hierarchy" collapses above the level $\mathcal{N}_3$;
2. For every formula $\varphi \in \mathcal{N}_2$ we have that $\mathbf{Int} + \varphi \in \{\mathbf{Form}, \mathbf{Int}\}$.

Theorem (Ciabattoni et al. 2008, 2017)

There is an effective procedure transforming any $\mathcal{P}_3$-axiom φ into a finite set of "equivalent" structural hypersequent rule $\mathscr{R}$ such that cut-admissibility is preserved when adding $\mathscr{R}$ to **HLJ**.

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Observation

At least countably many proper intermediate logics are axiomatisable by $\mathcal{P}_3$-formulas.

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At least countably many proper intermediate logics are axiomatisable by $\mathcal{P}_3$-formulas. E.g., $\mathbf{BW}_n, \mathbf{BTW}_n, \mathbf{BC}_n$, *for* $n \in \mathbb{N}$.

A problem with syntactic classifications

Given an intermediate logic $L := \mathbf{Int} + \varphi$ with $\varphi \notin \mathcal{P}_3$ there might exist $\psi \in \mathcal{P}_3$ such that $L = \mathbf{Int} + \psi$.

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$$\begin{aligned}
\mathbf{BTW}_n &= \mathbf{Int} + \bigwedge_{0 \leq i < j \leq n} \left(\neg(\neg p_i \wedge \neg p_j) \rightarrow \bigvee_{i=0}^{n} (\neg p_i \rightarrow \bigvee_{j \neq i} \neg p_j) \right) \\
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We need intrinsic semantic characterisations of logics with an $\mathcal{P}_3$-axiomatisation.

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Thus we only need to consider intermediate logics with a structural hypersequent calculus.

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Suppose that L admits a structural hypersequent calculus. Then, for Heyting algebras $\mathbf{A}, \mathbf{B}$,

$$\text{If } \mathbf{B} \in \mathbb{V}(L)_{si} \text{ and } \mathbf{A} \hookrightarrow_{0,\wedge,1} \mathbf{B} \text{ then } \mathbf{A} \in \mathbb{V}(L). \qquad (\dagger)$$

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If L is $(0, \wedge, 1)$-stable then $\mathbb{V}(L)$ is generated by a universal class of Heyting algebras axiomatised by a collection of universal $(0, \wedge, 1)$-clauses.

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$$\mathbf{B} \models q_{0,\wedge,1}(\mathbf{A}) \iff \mathbf{A} \not\hookrightarrow_{0,\wedge,1} \mathbf{B}.$$

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Corollary

Non of the logics $\mathbf{BD}_n$, for $n \geq 2$, can be captured by a structural extension of **HLJ**.

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where $\varphi_{ij}(\vec{w}, v)$ is either wRv or $w = v$ for some $w \in \vec{w}$.

Compare this with the *simple formulas* from (Lahav 2013).

Some corollaries

Let L be a intermediate logic with a structural hypersequent calculus extending **HLJ**. Then

1. L enjoys the finite model property;
2. L is a cofinal subframe logic;
3. L is Kripke complete;
4. The class of L-frames is an elementary class;
5. L is canonical;
6. L is axiomatised by $(\to, \wedge, \bot)$-formulas;
7. The class of well-connected $\mathbb{V}(L)$ algebras is closed under MacNeille completion;

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2. Can we do something similar for substructural and modal logics?
3. Are similar semantic characterisations available for other proof-theoretic formalisms?

Thank you very much for your time and attention.