Effect algebras as colimits of finite Boolean algebras
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Origins of the main idea of this talk

General idea

Let $E : \mathcal{C} \to \mathcal{D}$ be a functor. To avoid unnecessary problems, assume that \( \mathcal{C} \) is essentially small and \( \mathcal{D} \) is cocomplete.

Fix an object $A \in \mathcal{D}$. For every object $c \in \mathcal{C}$ we may consider the set of all \( \mathcal{D} \)-morphisms from $E(c)$ to $A$:

$c \mapsto \text{hom}_{\mathcal{D}}(E(c), A)$.

So every object of \( \mathcal{C} \) gives us an object of \( \text{Set} \).

This is a contravariant functor, for every $A \in \mathcal{D}$:

$A \mapsto \left[ \mathcal{C}^{\text{op}}, \text{Set} \right]$.

In other words, every object of \( \mathcal{D} \) induces a presheaf on \( \mathcal{C} \).
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- Fix an object $A \in D$.
- For every object $c \in C$ we may consider the set of all $D$-morphisms from $E(c)$ to $A$:
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General idea

- Moreover, the rule

\[ A \mapsto \mathcal{C}^{op}, \text{Set} \]

itself is functorial: morphism \( f : A \to A' \) in \( \mathcal{D} \) induces a natural transformation of presheaves in a straightforward way.
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- Thus, there is a functor \( R : \mathcal{D} \to [C^{op}, \text{Set}] \).
- It is a right adjoint functor.

The diagram shows the adjunction:

\[ [C^{op}, \text{Set}] \quad \perp \quad \mathcal{D} \]

\[ \begin{array}{c}
  \text{L} \\
  \downarrow \\
  \text{R}
\end{array} \]

For every object \( A \), the presheaf \( R(A) \) is something like "\( A \) from the point of view of \( C \)".

We may ask how much information about \( A \) is retained within \( R(A) \).

In case when the adjunction is a reflection, \( A \) can be reconstructed from \( R(A) \); \( E \) is then called dense.
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\[
\begin{array}{ccc}
[\text{C}^{\text{op}}, \text{Set}] & \nabla & D \\
\downarrow L \quad & \quad & \downarrow R \\
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\[ [C^{\text{op}}, \text{Set}] \xrightarrow{\perp} \mathcal{D} \xleftarrow{\perp} \]

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\[ \begin{array}{ccc}
  [C^{op}, \text{Set}] & & \mathcal{D} \\
  \downarrow & & \downarrow \\
  \mathcal{D} & & [C^{op}, \text{Set}] \\
  \mathcal{D} & \xleftarrow{R} & [C^{op}, \text{Set}] \\
  \mathcal{D} & \xrightarrow{L} & [C^{op}, \text{Set}] \\
\end{array} \]

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Effect algebras as colimits
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Effect algebras (the category $\mathcal{D}$)

An effect algebra \cite{Foulis1994, Kopka1994, Giuntini1989} is a partial algebra $(E; +, 0, 1)$ with a binary partial operation $+$ and two nullary operations $0, 1$ such that $+$ is commutative, associative and the following pair of conditions is satisfied:

\begin{itemize}
  \item[(E3)] For every $a \in E$ there is a unique $a' \in E$ such that $a + a'$ exists and $a + a' = 1$.
  \item[(E4)] If $a + 1$ is defined, then $a = 0$.
\end{itemize}

The $+$ operation is then cancellative and $0$ is a neutral element.
Effect algebras (the category $\mathcal{D}$)

The morphisms of effect algebras are defined in a natural way. By [Jacobs and Mandemaker(2012)], the category of effect algebras $\mathcal{EA}$ is complete and cocomplete. The category of effect algebras includes MV-algebras and orthomodular lattices as subcategories.
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- The category of effect algebras includes MV-algebras and orthomodular lattices as subcategories.
Boolean algebras are effect algebras

- Let $X$ be a Boolean algebra.
- Introduce a partial operation $+$ on $X$:
  - $a + b$ is defined iff $a \land b = 0$ and then $a + b = a \lor b$.
- $(X, +, 0, 1)$ is then an effect algebra.
- If $A$ is an effect algebra and $X$ is a Boolean algebra, a $\text{EA}$-morphism $X \to A$ is called an observable.
Let us consider the subcategory \( \text{FinBool} \) of \( E(A) \).

\( \text{FinBool} \) is essentially small.

\( \text{FinBool} \) is a full subcategory of \( E(A) \).

We understand morphisms in \( \text{FinBool} \) pretty well.

A morphism \( g \) from \( 2 \) into an effect algebra \( A \) is the same thing as a decomposition of \( 1 \in A \) into a sum of \( n \) elements of \( A \):

\[
g(\{1\}) + g(\{2\}) + \ldots + g(\{n\}) = 1.
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A morphism \( g \) from \( 2^{[n]} \) into an effect algebra \( A \) is the same thing as a decomposition of \( 1 \in A \) into a sum of \( n \) elements of \( A \)

\[
g(\{1\}) + g(\{2\}) + \cdots + g(\{n\}) = 1.
\]
Theorem

[Staton and Uijlen(2015)] The embedding $E : \text{FinBool} \rightarrow \text{EA}$ is dense. In particular, every effect algebra is a colimit of finite Boolean algebras (in a canonical way).
The category of elements of $R(A)$

For an effect algebra $A$, the category $\int R(A)$ is the category of finite observables, which can be explicitly described as follows:

- Objects are all pairs $(2^n, g)$, where $g : 2^n \to A$ is an observable.
- An arrow $(2^n, g) \to (2^{n'}, g')$ is a morphism of Boolean algebras $f : 2^n \to 2^{n'}$ such that $g' \circ f = g$. 

Effective algebras as colimits of finite Boolean algebras

Consider the functor

\[ D_A : \int R(A) \to \text{EA} \]

given by the rule

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Then

\[ \lim D_A = A \]
Riesz decomposition property

An effect algebra $A$ satisfies the Riesz decomposition property if and only if, for all $u, v_1, v_2 \in A$, $u \leq v_1 + v_2$ implies that there exist $u_1, u_2 \in A$ such that $u_1 \leq v_1$, $u_2 \leq v_2$ and $u = u_1 + u_2$.

We say that a category is amalgamated if and only if every span can be extended to a commutative square.

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**Theorem**

An effect algebra $A$ satisfies the Riesz decomposition property if and only if $\int R(A)$ is amalgamated.
An effect algebra $A$ is an orthoalgebra if, for all $a \in A$, $a \leq a'$ implies $a = 0$. 
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**Theorem**

An effect algebra $A$ is an orthoalgebra if and only if for every pair of morphisms $f_1, f_2 : g \to g'$ in $\int R(A)$ there is a coequalizing morphism $q : g' \to u$ such that $q \circ f_1 = q \circ f_2$. 
**Theorem**

An effect algebra $A$ is a Boolean algebra if and only if $\int R(A)$ is filtered.
Bimorphisms and tensor products

For effect algebras $A$, $B$ and $C$ a mapping $h : A \times B \to C$ is a $C$-valued bimorphism [Dvurečenskij(1995)] from $A$, $B$ to $C$ if and only if the following conditions are satisfied.

**Unitality:** $h(1,1) = 1$.

**Left additivity:** For all $b \in B$ and $a_1, a_2 \in A$ such that $a_1 \perp a_2$,
\[ h(a_1, b) \perp h(a_2, b) \text{ and } h(a_1, b) + h(a_2, b) = h(a_1 + a_2, b). \]

**Right additivity:** For all $a \in A$ and $b_1, b_2 \in B$ such that $b_1 \perp b_2$,
\[ h(a, b_1) \perp h(a, b_2) \text{ and } h(a, b_1) + h(a, b_2) = h(a, b_1 + b_2). \]
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$h(a, b_1) \perp h(a, b_2)$ and $h(a, b_1) + h(a, b_2) = h(a, b_1 + b_2)$.

There is a category $\beta_{A,B}$ where the objects are all bimorphisms from $A, B$ and the morphisms are $\text{EA}$-morphisms acting on bimorphisms from left by composition.
Definition

[Dvurečenskij(1995)] Let $A, B$ be effect algebras. A tensor product of $A$ and $B$ (denoted by $A \otimes B$) is the initial object in the category $\beta_{A,B}$.

$$A \times B \rightarrow A \otimes B$$
Let $A$, $B$ be effect algebras. The category $\int \mathbb{R}(A) \times \int \mathbb{R}(B)$ has pairs of finite observables as objects and pairs of morphisms of observables as arrows. Consider the functor $D_{A,B}: \int \mathbb{R}(A) \times \int \mathbb{R}(B) \to \mathcal{E}A$ given by the rule $D_{A,B}(g_A, g_B) = \text{Dom}(g_A)^* \text{Dom}(g_B)$, where $^*$ denotes free product (that means, coproduct in $\mathbb{Bool}$).
Tensor products as colimits

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Consider the functor $D_{A,B} : \mathcal{I} R(A) \times \mathcal{I} R(B) \to \mathbf{EA}$ given by the rule

$$D_{A,B}(g_A, g_B) = \text{Dom}(g_A) \ast \text{Dom}(g_B),$$

where $\ast$ denotes free product (that means, coproduct in $\mathbf{Bool}$) of Boolean algebras.
Lemma

Let $A, B$ be effect algebras. The category of bimorphisms $\beta_{A,B}$ is isomorphic to the category of cocones under the diagram $D_{A,B}$. Under this isomorphism, $C$-valued bimorphisms correspond to cocones with apex $C$ and vice versa.
Lemma
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Corollary
For every pair $A, B$ of effect algebras,

$$A \otimes B = \lim_{\rightarrow} D_{A,B}$$
Theorem

The tensor product of effect algebras is a functor $EA \times EA \to EA$ that arises as a left Kan extension of the functor $E \circ \ast : \text{FinBool} \times \text{FinBool} \to EA$ along the inclusion $E \times E : \text{FinBool} \times \text{FinBool} \to EA \times EA$.

$EA \times EA \otimes \to \to \text{FinBool} \times \text{FinBool}$

$E \times E \uparrow \uparrow \ast \to \to \text{FinBool}$

$E \to \to \uparrow \uparrow \eta_{EA}(1)$
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(1)
Tensor products from Day convolution

It was proved by Day in [Day(1970)] that for every monoidal category $(\mathcal{C}, \boxtimes, I)$, the monoidal structure can be extended to the category $[\mathcal{C}^{op}, \text{Set}]$ of presheaves on $\mathcal{C}$ by the rule

$$X \otimes_{\text{Day}} Y = \int (c_1, c_2) \mathcal{C}^{op}(c_1 \boxtimes c_2, c_2) \times X(c_1) \times Y(c_2).$$
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**Theorem**

For every pair \(A, B\) of effect algebras,

\[ A \otimes B \simeq L(R(A) \otimes_{\text{Day}} R(B)) \]
Brian Day.  
On closed categories of functors, pages 1–38.  
ISBN 978-3-540-36292-0.  
doi: 10.1007/BFb0060438.  
URL http://dx.doi.org/10.1007/BFb0060438.

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