Effect algebras as colimits of finite Boolean algebras

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Origins of the main idea of this talk

Staton, S., Uijlen, S.: Effect algebras, presheaves, non-locality and

contextuality. In: International Colloquium on Automata, Languages, and Programming, pp. 401–413. Springer (2015)

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- This is a contravariant functor, for every $A \in \mathcal{D}$:
$$A \mapsto [\mathcal{C}^{op}, \mathbf{Set}]$$
- In other words, every object of $\mathcal{D}$ induces a presheaf on $\mathcal{C}$.

General idea

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- For every object A, the presheaf $R(A)$ is something like "A from the point of view of $\mathcal{C}$".
- We may ask how much information about A is retained within $R(A)$.
- In case when the adjunction is a reflection, A can be reconstructed from $R(A)$; E is then called dense.

Effect algebras (the category $\mathcal{D}$)

An effect algebra [Foulis and Bennett(1994), Kôpka and Chovanec(1994), Giuntini and Greuling(1989)] is a partial algebra $(E; +, 0, 1)$ with a binary partial operation $+$ and two nullary operations $0, 1$ such that $+$ is commutative, associative and the following pair of conditions is satisfied:

(E3) For every $a \in E$ there is a unique $a' \in E$ such that $a + a'$ exists and $a + a' = 1$.

(E4) If $a + 1$ is defined, then $a = 0$.

The $+$ operation is then cancellative and 0 is a neutral element.

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- By [Jacobs and Mandemaker(2012)], the category of effect algebras **EA** is complete and cocomplete.
- The category of effect algebras includes MV-algebras and orthomodular lattices as subcategories.

Boolean algebras are effect algebras

- Let X be a Boolean algebra.
- Introduce a partial operation $+$ on X:
- $a + b$ is defined iff $a \wedge b = 0$ and then $a + b = a \vee b$.
- $(X, +, 0, 1)$ is then an effect algebra.
- If A is an effect algebra and X is a Boolean algebra, a **EA**-morphism $X \to A$ is called an observable.

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- Let us consider the subcategory **FinBool** of **EA**.
- **FinBool** is essentially small.
- **FinBool** is a full subcategory of **EA**.
- We understand morphisms in **FinBool** pretty well.
- A morphism g from $2^{[n]}$ into an effect algebra A is the same thing as a decomposition of $1 \in A$ into a sum of n elements of A

$$g(\{1\}) + g(\{2\}) + \cdots + g(\{n\}) = 1.$$

Theorem

[Staton and Uijlen(2015)] The embedding $E : \mathbf{FinBool} \to \mathbf{EA}$ *is dense. In particular, every effect algebra is a colimit of finite Boolean algebras (in a canonical way).*

The category of elements of $R(A)$

For an effect algebra A, the category $\int R(A)$ is the <u>category of finite observables</u>, which can be explicitly described as follows:

- Objects are all pairs $(2^{[n]}, g)$, where $g : 2^{[n]} \to A$ is an observable.
- An arrow $(2^{[n]}, g) \to (2^{[n']}, g')$ is a morphism of Boolean algebras $f : 2^{[n]} \to 2^{[n']}$ such that $g' \circ f = g$.

Effect algebras as colimits of finite Boolean algebras

Consider the functor

$$D_A : \int R(A) \to \mathbf{EA}$$

given by the rule

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Then

$$\varinjlim D_A = A$$

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- An effect algebra A satisfies the Riesz decomposition property if and only if, for all $u, v_1, v_2 \in A$, $u \leq v_1 + v_2$ implies that there exist $u_1, u_2 \in A$ such that $u_1 \leq v_1$, $u_2 \leq v_2$ and $u = u_1 + u_2$.

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Theorem

An effect algebra A satisfies the Riesz decomposition property if and only if $\int R(A)$ is amalgamated.

Orthoalgebras

An effect algebra A is an orthoalgebra if, for all $a \in A$, $a \leq a'$ implies $a = 0$.

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Theorem

An effect algebra A is an orthoalgebra if and only if for every pair of morphisms $f_1, f_2 : g \to g'$ in $\int R(A)$ there is a coequalizing morphism $q : g' \to u$ such that $q \circ f_1 = q \circ f_2$.

Boolean algebras

Theorem

An effect algebra A is a Boolean algebra if and only if $\int R(A)$ is filtered.

Bimorphisms and tensor products

For effect algebras A, B and C a mapping $h : A \times B \to C$ is a C-valued bimorphism [Dvurečenskij(1995)] from A, B to C if and only if the following conditions are satisfied.

Unitality: $h(1, 1) = 1$.

Left additivity: For all $b \in B$ and $a_1, a_2 \in A$ such that $a_1 \perp a_2$, $h(a_1, b) \perp h(a_2, b)$ and $h(a_1, b) + h(a_2, b) = h(a_1 + a_2, b)$.

Right additivity: For all $a \in A$ and $b_1, b_2 \in B$ such that $b_1 \perp b_2$, $h(a, b_1) \perp h(a, b_2)$ and $h(a, b_1) + h(a, b_2) = h(a, b_1 + b_2)$.

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There is a category $\beta_{A,B}$ where the objects are all bimorphisms from A, B and the morphisms are **EA**-morphisms acting on bimorphisms from left by composition.

Definition

[Dvurečenskij(1995)] Let A, B be effect algebras. A tensor product of A and B (denoted by $A \otimes B$) is the initial object in the category $\beta_{A,B}$.

$$A \times B \xrightarrow{\otimes} A \otimes B$$

Tensor products as colimits

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- The category $\int R(A) \times \int R(B)$ has pairs of finite observables as objects and pairs of morphisms of observables as arrows.
- Consider the functor $D_{A,B} : \int R(A) \times \int R(B) \to \mathbf{EA}$ given by the rule

$$D_{A,B}(g_A, g_B) = \mathrm{Dom}(g_A) * \mathrm{Dom}(g_B),$$

where $*$ denotes free product (that means, coproduct in **Bool**) of Boolean algebras.

Tensor products as colimits

Lemma

Let A, B be effect algebras. The category of bimorphisms $\beta_{A,B}$ is isomorphic to the category of cocones under the diagram $D_{A,B}$. Under this isomorphism, C-valued bimorphisms correspond to cocones with apex C and vice versa.

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Corollary

For every pair A, B of effect algebras,

$$A \otimes B = \varinjlim D_{A,B}$$

Tensor products as left Kan extensions

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Theorem

The tensor product of effect algebras is a functor $\mathbf{EA} \times \mathbf{EA} \to \mathbf{EA}$ *that arises as a left Kan extension of the functor* $E \circ * : \mathbf{FinBool} \times \mathbf{FinBool} \to \mathbf{EA}$ *along the inclusion* $E \times E : \mathbf{FinBool} \times \mathbf{FinBool} \to \mathbf{EA} \times \mathbf{EA}$.

$$\begin{tikzcd} \mathbf{EA} \times \mathbf{EA} \arrow[drr, "\otimes"] & & \\ \mathbf{FinBool} \times \mathbf{FinBool} \arrow[u, "E \times E"] \arrow[r, "*"'] \arrow[ur, phantom, "\eta \Uparrow"] & \mathbf{FinBool} \arrow[r, "E"'] & \mathbf{EA} \end{tikzcd} \tag{1}$$

Tensor products from Day convolution

It was proved by Day in [Day(1970)] that for every monoidal category $(\mathcal{C}, \square, I)$, the monoidal structure can be extended to the category $[\mathcal{C}^{op}, \mathbf{Set}]$ of presheaves on $\mathcal{C}$ by the rule

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Theorem

For every pair A, B of effect algebras,

$$A \otimes B \simeq L(R(A) \otimes_{\mathrm{Day}} R(B))$$

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