Multi-type Display Calculus for Semi-De Morgan Logic

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Motivation and Aim

- Sankappanavar H P. Semi-De Morgan algebras[J]. *The Journal of symbolic logic*, 1987, 52(3): 712-724
- a common abstraction of De Morgan algebras and distributive pseudo-complemented lattices

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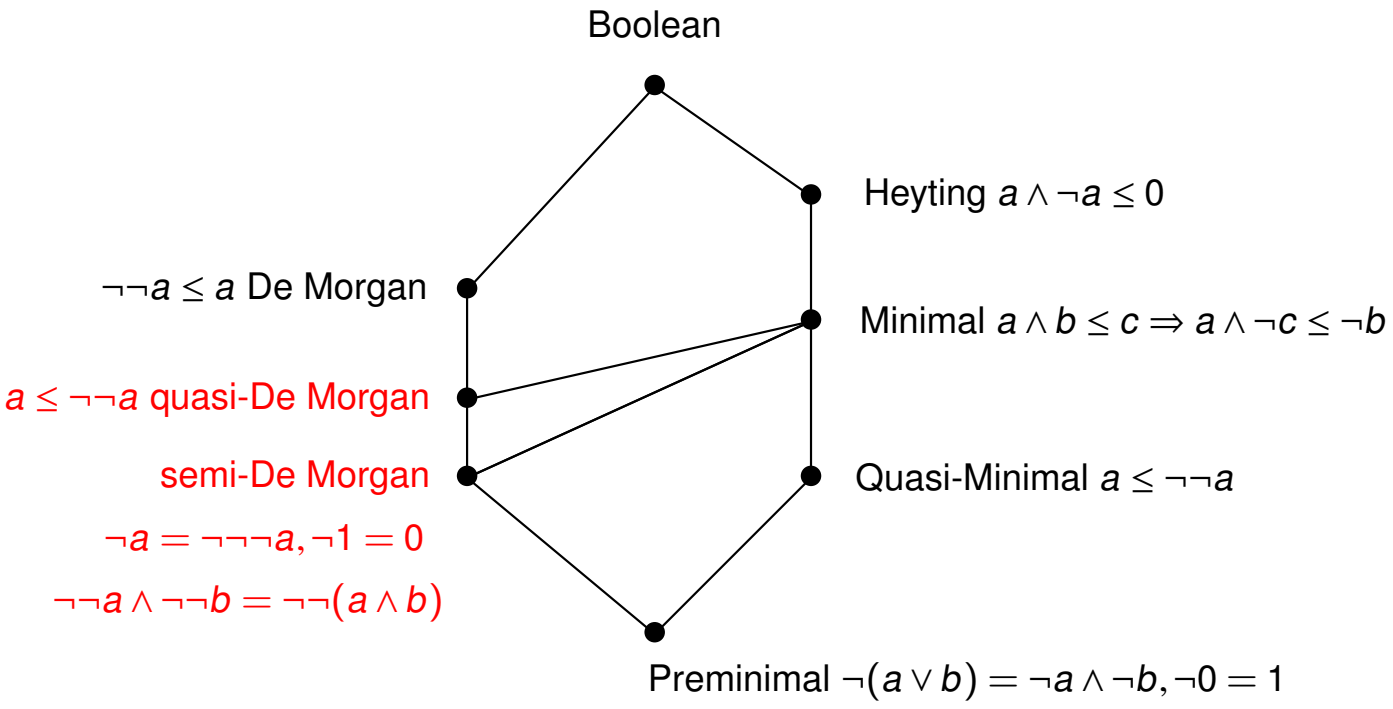

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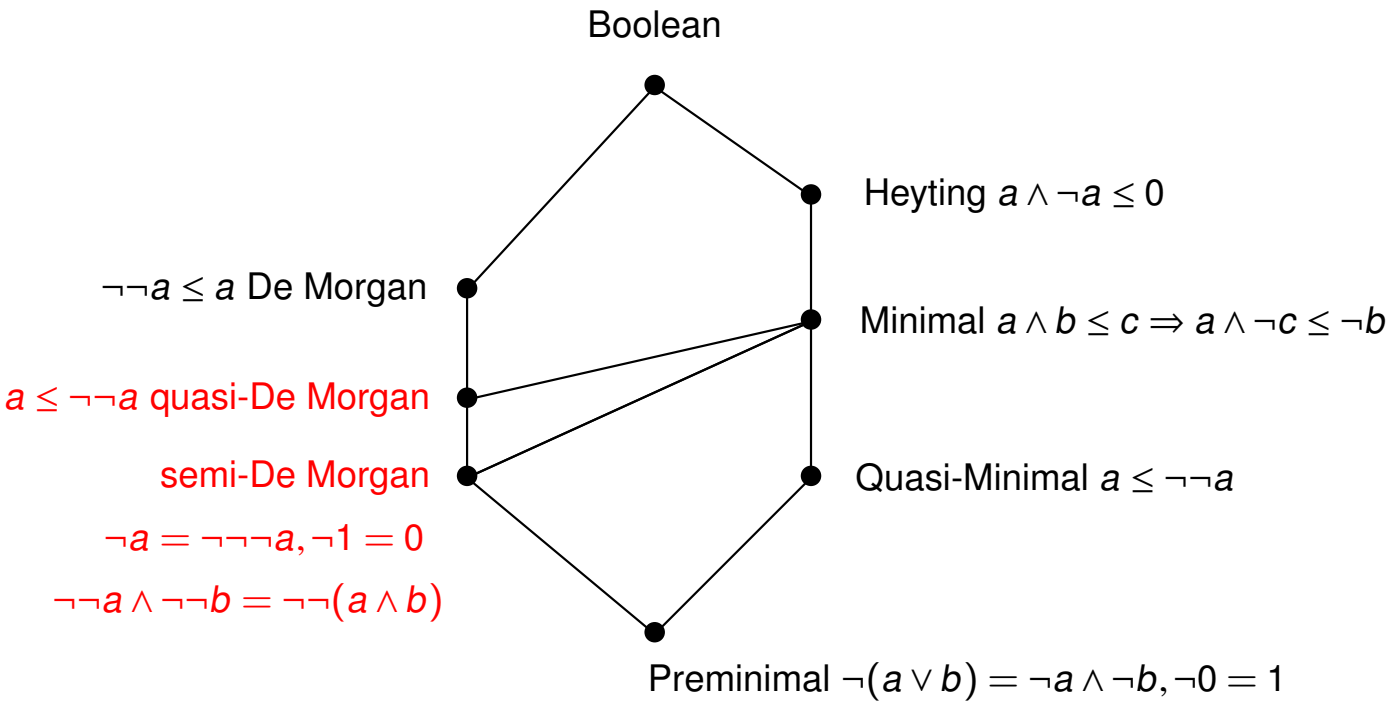

- Ma M, Liang F. Sequent Calculi for Semi-De Morgan and De Morgan Algebras[J]. *arXiv preprint:1611.05231*, 2016.

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Is there an uniform way to deal with semi De Morgan negation and preserve real subformula property?

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- The answer is “Yes”, via **multi-type methodology**!

Preliminaries

De Morgan and semi-De Morgan Algebras

Definition

If $(A, \vee, \wedge, \top, \perp)$ is a bounded distributive lattice, then an algebra $\mathfrak{A} = (A, \vee, \wedge, \neg, \top, \perp)$ is: for all $a, b \in A$,

De Morgan algebra	semi-De Morgan algebra
$\neg(a \vee b) = \neg a \wedge \neg b$	$\neg(a \vee b) = \neg a \wedge \neg b$
$\neg(a \wedge b) = \neg a \vee \neg b$	$\neg\neg(a \wedge b) = \neg\neg a \wedge \neg\neg b$
$\neg\neg a = a$	$\neg\neg\neg a = \neg a$
$\neg\perp = \top, \neg\top = \perp$	$\neg\perp = \top$, $\neg\top = \perp$

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Fact

A semi-De Morgan algebra $\mathfrak{A}$ is a De Morgan algebra if and only if $\mathfrak{A}$ satisfies the equation $a \vee b = \neg(\neg a \wedge \neg b) = \neg\neg(a \vee b)$.

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$\neg\neg(a \wedge b) = \neg\neg a \wedge \neg\neg b$ and $\neg\neg\neg a = \neg a$ can not be transformed into structural rules immediately!

Stratergy

- from semi-De Morgan algebras to construct heterogeneous semi-De Morgan algebras in which every axiom is analytic

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From single type to multi-type

Multi-type enviroment

Lemma

Given an SM-algebra $\mathbb{L} = (L, \wedge, \vee, \top, \bot, \neg)$*, let* $K := \{\neg\neg a \in L \mid a \in L\}$*. Define* $h : L \twoheadrightarrow K$ *and* $e : K \hookrightarrow L$ *by the assignments* $a \mapsto \neg\neg a$ *and* $\alpha \mapsto \alpha$*, respectively. Then for all* $\alpha \in K$ *and* $a \in L$*,*

$$h(e(\alpha)) = \alpha$$

Multi-type enviroment

Definition

For any SM-algebra $\mathbb{L} = (L, \wedge, \vee, \top, \bot, \neg)$, let the *kernel* of $\mathbb{L}$ be the algebra $\mathbb{K}_{\mathbb{L}} = (K, \cap, \cup, \sim, 1, 0)$ defined as follows:

K1. $K := Range(h)$, where $h : L \twoheadrightarrow K$ is defined by letting $h(a) = \neg\neg a$ for any $a \in L$;

K2. $\alpha \cup \beta := h(\neg\neg(e(\alpha) \vee e(\beta)))$ for all $\alpha, \beta \in K$;

K3. $\alpha \cap \beta := h(e(\alpha) \wedge e(\beta))$ for all $\alpha, \beta \in K$;

K4. $1 := h(\top)$;

K5. $0 := h(\bot)$;

K6. $\sim\alpha := h(\neg e(\alpha))$.

Multi-type enviroment

Lemma

For any SM-algebra $\mathbb{L}$,

1. *the kernel $\mathbb{K}_{\mathbb{L}}$ is a DM-algebra.*
2. *h is a lattice-homomorphism from $\mathbb{L}$ onto $\mathbb{K}$, and for all $\alpha,\beta \in K$,*

$$e(\alpha) \wedge e(\beta) = e(\alpha \cap \beta) \qquad e(1) = \top \qquad e(0) = \bot.$$

Heterogenous algebra

Definition

A *heterogeneous SDM-algebra* (HSM-algebra) is a tuple $(\mathbb{L}, \mathbb{A}, e, h)$ satisfying the following conditions:

H1 $\mathbb{L}$ is a bounded distributive lattice;

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H2 $\mathbb{A}$ is a De Morgan lattice;

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H1 $\mathbb{L}$ is a bounded distributive lattice;

H2 $\mathbb{A}$ is a De Morgan lattice;

H3 $e : \mathbb{A} \hookrightarrow \mathbb{L}$ is an order embedding, which satisfies: for all $\alpha_1, \alpha_2 \in \mathbb{A}$,

$$e(\alpha_1) \wedge e(\alpha_2) = e(\alpha_1 \cap \alpha_2) \quad \text{and} \quad e(1) = \top \quad \text{and} \quad e(0) = \bot$$

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H4 $h : \mathbb{L} \twoheadrightarrow \mathbb{A}$ is a lattice homomorphism;

H5 $h(e(\alpha)) = \alpha$ for every $\alpha \in \mathbb{A}$.

$$\begin{tikzcd}
\mathbb{L} \arrow[rr, two heads, "h"'] & & \mathbb{A} \arrow[ll, bend right, "e"'] \arrow[loop below, "\sim"]
\end{tikzcd}$$

From multi-type to single type

Heterogenous algebra

Lemma

If $(\mathbb{L}, \mathbb{D}, e, h)$ *is an heterogeneous SM-algebra, then* $\mathbb{L}$ *can be endowed with a structure of SM-algebra defining* $\neg : \mathbb{L} \to \mathbb{L}$ *by* $\neg a := e(\sim h(a))$ *for every* $a \in \mathbb{L}$*. Moreover,* $\mathbb{D} \cong \mathbb{K}$*.*

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Definition

For any SM-algebra $\mathbb{A}$, we let $\mathbb{A}^+ = (\mathbb{L}, \mathbb{K}, h, e)$, where:

- $\mathbb{L}$ is the lattice reduct of $\mathbb{A}$;
- $\mathbb{K}$ is the kernel of $\mathbb{A}$;
- $e : \mathbb{K} \hookrightarrow \mathbb{L}$ is defined by $e(\alpha) = \alpha$ for all $\alpha \in \mathbb{K}$;
- $h : \mathbb{L} \twoheadrightarrow \mathbb{K}$ is defined by $h(a) = \neg\neg a$ for all $a \in \mathbb{L}$;

For any HSM-algebra $\mathbb{H}$, we let $\mathbb{H}_+ = (\mathbb{L}, \neg)$ where:

- $\mathbb{L}$ is the distributive lattice of $\mathbb{H}$;
- $\neg : \mathbb{L} \to \mathbb{L}$ is defined by the assignment $a \mapsto e(\sim h(a))$ for all $a \in \mathbb{L}$.

Heterogenous representation theory

For any SM-algebra $\mathbb{A}$ and any HSM-algebra $\mathbb{H}$:

$$\mathbb{A} \cong (\mathbb{A}^+)_+ \quad \text{and} \quad \mathbb{H} \cong (\mathbb{H}_+)^+.$$

Algebraic semantics for multi-type display calculus

Canonical extension

Definition

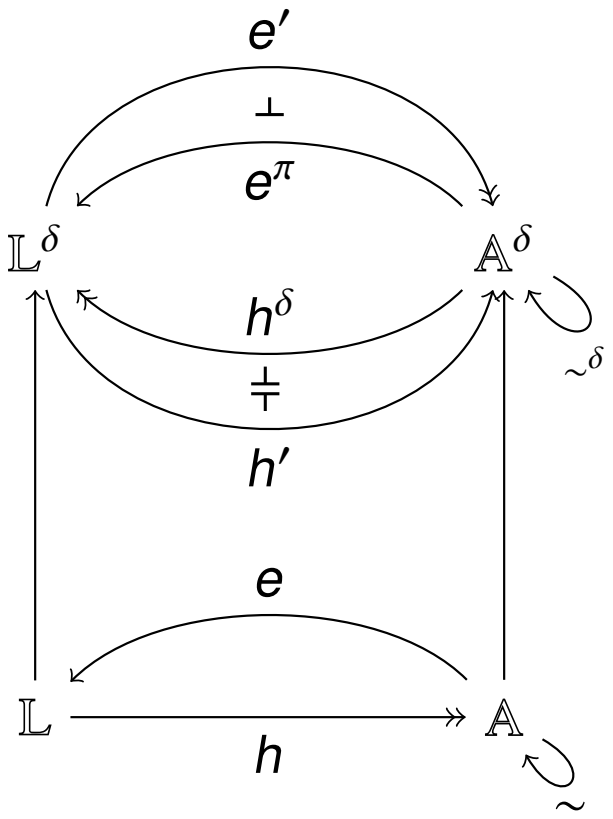
A HSM-algebra is *perfect* if:

1. both $\mathbb{L}$ and $\mathbb{A}$ are perfect;
2. e is an order-embedding and is completely meet-preserving;
3. h is a complete homomorphism.

Corollary

If $(\mathbb{L}, \mathbb{D}, e, h)$ *is an HSM-algebra, then* $(\mathbb{L}^\delta, \mathbb{D}^\delta, e^\pi, h^\delta)$ *is a perfect HSM-algebra.*

Canonical extension



Corollary

If $(\mathbb{L}, \neg)$ is an SM-algebra, then $\mathbb{L}^\delta$ can be endowed with the structure of SM-algebra by defining $\neg^\delta : \mathbb{L}^\delta \to \mathbb{L}^\delta$ by $\neg^\delta := e^\pi \circ \sim^\delta \circ h^\delta$. Moreover, $\mathbb{K}^\delta_{\mathbb{L}} \cong \mathbb{K}_{\mathbb{L}^\delta}$.

Multi-type proper display calculus

Hilbert style semi-De Morgan logic

- the language $\mathcal{L}$

$$A ::= p \mid \bot \mid \top \mid \neg A \mid A \wedge A \mid A \vee A$$

- Axioms

(A1)	$\bot \vdash A$	(A2)	$A \vdash \top$
(A3)	$\neg\top \vdash \bot$	(A4)	$\top \vdash \neg\bot$
(A5)	$A \vdash A$	(A6)	$A \wedge B \vdash A$
(A7)	$A \wedge B \vdash B$	(A8)	$A \vdash A \vee B$
(A9)	$B \vdash A \vee B$	(A10)	$\neg A \vdash \neg\neg\neg A$
(A11)	$\neg\neg\neg A \vdash \neg A$		
(A12)	$\neg A \wedge \neg B \vdash \neg(A \vee B)$		
(A13)	$\neg\neg A \wedge \neg\neg B \vdash \neg\neg(A \wedge B)$		
(A14)	$A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$		

- Rules

R1. If $A \vdash B$ and $B \vdash C$, then $A \vdash C$;

R2. If $A \vdash B$ and $A \vdash C$, then $A \vdash B \wedge C$;

R3. If $A \vdash B$ and $C \vdash B$, then $A \vee C \vdash B$;

R4. If $A \vdash B$, then $\neg B \vdash \neg A$.

Multi-type Display calculus

- Structural and operational language of D.DL:

$$DL \begin{cases} A ::= p \mid \top \mid \bot \mid \Box\alpha \mid A \wedge A \mid A \vee A \\ \\ X ::= \hat{\top} \mid \check{\bot} \mid \check{\Box}\Gamma \mid X \hat{\wedge} X \mid X \check{\vee} X \mid X \succ\!\!\!\!- X \mid X \check{\rightarrow} X \end{cases}$$

Multi-type Display calculus

- Structural and operational language of D.DL:

$$DL \left\{ \begin{array}{l} A ::= p \mid \top \mid \bot \mid \Box\alpha \mid A \wedge A \mid A \vee A \\ \\ X ::= \hat{\top} \mid \check{\bot} \mid \check{\Box}\Gamma \mid X \mathbin{\hat{\wedge}} X \mid X \mathbin{\check{\vee}} X \mid X \mathbin{\hat{\succ\!\!-}} X \mid X \mathbin{\check{\rightarrow}} X \end{array} \right.$$

- Structural and operational language of D.DM:

$$DM \left\{ \begin{array}{l} \alpha ::= \circ A \mid 1 \mid 0 \mid {\sim}\alpha \mid \alpha \cap \alpha \mid \alpha \cup \alpha \\ \\ \Gamma ::= \tilde{\circ} X \mid \hat{1} \mid \check{0} \mid \tilde{\sim}\Gamma \mid \Gamma \mathbin{\hat{\cap}} \Gamma \mid \Gamma \mathbin{\check{\cup}} \Gamma \mid \Gamma \mathbin{\hat{\succ\!\!-}_{\neg}} \Gamma \mid \Gamma \mathbin{\check{\rightarrow}_{\neg}} \Gamma \end{array} \right.$$

Interpretation

- Interpretation of structural *DL* connectives as their operational counterparts

DL connectives

categorization	f			g		
structural	$\hat{\top}$	$\hat{\wedge}$	$\hat{\succ\!\!-}$	$\check{\bot}$	$\check{\vee}$	$\check{\rightarrow}$
operational	$\top$	$\wedge$	$(\succ\!\!-)$	$\bot$	$\vee$	$(\rightarrow)$
adjoint pairs	$\hat{\wedge} \dashv \check{\rightarrow}$			$\hat{\succ\!\!-} \dashv \check{\vee}$		

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adjoint pairs	$\hat{\wedge} \dashv \check{\rightarrow}$			$\hat{\succ\!\!-} \dashv \check{\vee}$		

- Interpretation of structural *DM* connectives as their operational counterparts

DM connectives

categorization	f			g			f-g
structural	$\hat{1}$	$\hat{\cap}$	$\hat{\succ\!\!-}_{\neg}$	$\check{0}$	$\check{\cup}$	$\check{\rightarrow}_{\neg}$	$\tilde{\sim}$
operational	1	$\cap$	$(\succ\!\!-_{\neg})$	0	$\cup$	$(\rightarrow_{\neg})$	$\sim$
adjoint pairs	$\hat{\cap} \dashv \check{\rightarrow}_{\neg}$			$\hat{\succ\!\!-}_{\neg} \dashv \check{\cup}$			$\tilde{\sim} \dashv \tilde{\sim}$

Interpretation

- Interpretation of structural heterogeneous (from *DL* to *DM* and vice versa) connectives as their operational counterparts

	$DL \to DM$	$DM \to DL$	$DM \to DL$	$DL \to DM$
categorization	f-g	f-g	g	f
structural	$\tilde{\circ}$	$\tilde{\bullet}$	$\check{\square}$	$\hat{\blacklozenge}$
operational	$\circ$	$\bullet$	$\square$	$\blacklozenge$
adjoint pairs	$\tilde{\circ} \dashv \tilde{\bullet}$		$\hat{\blacklozenge} \dashv \check{\square}$	

Display Postulates

- DL-type display structural rules

$$\text{res}\ \frac{X \mathbin{\hat{\wedge}} Y \vdash Z}{Y \vdash X \mathbin{\check{\rightarrow}} Z} \qquad \frac{X \vdash Y \mathbin{\check{\vee}} Z}{Y \mathbin{\hat{\succ\!\!-}} X \vdash Z}\ \text{res}$$

Display Postulates

- DL-type display structural rules

$$\text{res}\ \frac{X \mathbin{\hat{\wedge}} Y \vdash Z}{Y \vdash X \mathbin{\check{\to}} Z} \qquad \frac{X \vdash Y \mathbin{\check{\vee}} Z}{Y \mathbin{\hat{\succ\!\!-}} X \vdash Z}\ \text{res}$$

- De Morgan lattice type display structural rules

$$\text{adj}\ \frac{\mathbin{\tilde{\sim}}\Gamma \vdash \Delta}{\mathbin{\tilde{\sim}}\Delta \vdash \Gamma} \qquad \frac{\Gamma \vdash \mathbin{\tilde{\sim}}\Delta}{\Delta \vdash \mathbin{\tilde{\sim}}\Gamma}\ \text{adj}$$

$$\text{res}\ \frac{\Gamma \mathbin{\hat{\cap}} \Delta \vdash \Theta}{\Delta \vdash \Gamma \mathbin{\check{\to}}_{\neg} \Theta} \qquad \frac{\Gamma \vdash \Delta \mathbin{\check{\cup}} \Theta}{\Delta \mathbin{\hat{\succ\!\!-}}_{\neg} \Gamma \vdash \Theta}\ \text{res}$$

DL-type structural rules

$$\mathsf{Id}\ \frac{}{p \vdash p} \qquad \frac{X \vdash A \qquad A \vdash Y}{X \vdash Y}\ \mathsf{Cut}$$

$$\hat{\top}\ \frac{X \vdash Y}{X \mathbin{\hat{\wedge}} \hat{\top} \vdash Y} \qquad \frac{X \vdash Y}{X \vdash Y \mathbin{\check{\vee}} \check{\bot}}\ \check{\bot}$$

$$\mathsf{E}\ \frac{X \mathbin{\hat{\wedge}} Y \vdash Z}{Y \mathbin{\hat{\wedge}} X \vdash Z} \qquad \frac{X \vdash Y \mathbin{\check{\vee}} Z}{X \vdash Z \mathbin{\check{\vee}} Y}\ \mathsf{E}$$

$$\mathsf{A}\ \frac{(X \mathbin{\hat{\wedge}} Y) \mathbin{\hat{\wedge}} Z \vdash W}{X \mathbin{\hat{\wedge}} (Y \mathbin{\hat{\wedge}} Z) \vdash Z} \qquad \frac{X \vdash (Y \mathbin{\check{\vee}} Z) \mathbin{\check{\vee}} W}{X \vdash Y \mathbin{\check{\vee}} (Z \mathbin{\check{\vee}} W)}\ \mathsf{A}$$

$$\mathsf{W}\ \frac{X \vdash Y}{X \mathbin{\hat{\wedge}} Z \vdash Y} \qquad \frac{X \vdash Y}{X \vdash Y \mathbin{\check{\vee}} Z}\ \mathsf{W}$$

$$\mathsf{C}\ \frac{X \mathbin{\hat{\wedge}} X \vdash Y}{X \vdash Y} \qquad \frac{X \vdash Y \mathbin{\check{\vee}} Y}{X \vdash Y}\ \mathsf{C}$$

DM-type structural rules

$$\frac{\Gamma \vdash \alpha \qquad \alpha \vdash \Delta}{\Gamma \vdash \Delta}\ \mathsf{Cut}$$

$$\hat{1}\ \frac{\Gamma \vdash \Delta}{\Gamma \mathbin{\hat{\cap}} \hat{1} \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta \mathbin{\check{\cup}} \check{0}}\ \check{0}$$

$$\mathsf{E}\ \frac{\Gamma \mathbin{\hat{\cap}} \Delta \vdash \Theta}{\Delta \mathbin{\hat{\cap}} \Gamma \vdash \Theta} \qquad \frac{\Gamma \vdash \Delta \mathbin{\check{\cup}} \Theta}{\Gamma \vdash \Theta \mathbin{\check{\cup}} \Delta}\ \mathsf{E}$$

$$\mathsf{A}\ \frac{(\Gamma \mathbin{\hat{\cap}} \Delta) \mathbin{\hat{\cap}} \Theta \vdash \Lambda}{\Gamma \mathbin{\hat{\cap}} (\Delta \mathbin{\hat{\cap}} \Theta) \vdash \Lambda} \qquad \frac{\Gamma \vdash (\Delta \mathbin{\check{\cup}} \Theta) \mathbin{\check{\cup}} \Lambda}{\Gamma \vdash \Delta \mathbin{\check{\cup}} (\Theta \mathbin{\check{\cup}} \Lambda)}\ \mathsf{A}$$

$$\mathsf{W}\ \frac{\Gamma \vdash \Delta}{\Gamma \mathbin{\hat{\cap}} \Theta \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta \mathbin{\check{\cup}} \Theta}\ \mathsf{W}$$

$$\mathsf{C}\ \frac{\Gamma \mathbin{\hat{\cap}} \Gamma \vdash \Delta}{\Gamma \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta \mathbin{\check{\cup}} \Delta}{\Gamma \vdash \Delta}\ \mathsf{C}$$

$$\frac{\Gamma \vdash \Delta}{\sim\Delta \vdash \sim\Gamma}\ \mathsf{cont}$$

DL-type operational rules

$$\top\ \frac{\hat{\top} \vdash X}{\top \vdash X} \qquad \frac{}{\hat{\top} \vdash \top}\ \top$$

$$\bot\ \frac{}{\bot \vdash \check{\bot}} \qquad \frac{X \vdash \check{\bot}}{X \vdash \bot}\ \bot$$

$$\wedge\ \frac{A \mathbin{\hat{\wedge}} B \vdash X}{A \wedge B \vdash X} \qquad \frac{X \vdash A \qquad Y \vdash B}{X \mathbin{\hat{\wedge}} Y \vdash A \wedge B}\ \wedge$$

$$\vee\ \frac{A \vdash X \qquad B \vdash Y}{A \vee B \vdash X \mathbin{\check{\vee}} Y} \qquad \frac{X \vdash A \mathbin{\check{\vee}} B}{X \vdash A \vee B}\ \vee$$

DM-type operational rules

$$1\ \frac{\hat{1} \vdash \Gamma}{1 \vdash \Gamma} \qquad \frac{}{\hat{1} \vdash 1}\ 1$$

$$0\ \frac{}{0 \vdash \check{0}} \qquad \frac{\Gamma \vdash \check{0}}{\Gamma \vdash 0}\ 0$$

$$\cap\ \frac{\alpha \hat{\cap} \beta \vdash \Gamma}{\alpha \cap \beta \vdash \Gamma} \qquad \frac{\Gamma \vdash \alpha \qquad \Delta \vdash \beta}{\Gamma \hat{\cap} \Delta \vdash \alpha \cap \beta}\ \cap$$

$$\cup\ \frac{\alpha \vdash \Gamma \qquad \beta \vdash \Delta}{\alpha \cup \beta \vdash \Gamma \check{\cup} \Delta} \qquad \frac{\Gamma \vdash \alpha \check{\cup} \beta}{\Gamma \vdash \alpha \cup \beta}\ \cup$$

$$\sim\ \frac{\tilde{\sim}\alpha \vdash \Gamma}{\sim\alpha \vdash \Gamma} \qquad \frac{\Gamma \vdash \tilde{\sim}\alpha}{\Gamma \vdash \sim\alpha}\ \sim$$

Multi-type rules

- Multi-type display postulates

$$\text{adj}\ \frac{X \vdash \check{\Box}\Gamma}{\hat{\blacklozenge}X \vdash \Gamma} \qquad \frac{\tilde{\circ}X \vdash \Gamma}{X \vdash \tilde{\bullet}\Gamma}\ \text{adj}$$

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- Multi-type structural rules

$$\tilde{\circ}\ \frac{X \vdash Y}{\tilde{\circ}X \vdash \tilde{\circ}Y} \qquad \frac{\Gamma \vdash \tilde{\circ}\check{\Box}\Delta}{\Gamma \vdash \Delta}\ \tilde{\circ}\check{\Box}$$

$$\check{\Box}\hat{1}\ \frac{X \vdash \check{\Box}\hat{1}}{X \vdash \hat{\top}} \qquad \frac{X \vdash \check{\Box}\check{0}}{X \vdash \check{\bot}}\ \check{\Box}\check{0}$$

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- Multi-type operational rules

$$\circ\ \frac{\tilde{\circ}A \vdash Y}{\circ A \vdash Y} \qquad \frac{X \vdash \tilde{\circ}A}{X \vdash \circ A}\ \circ$$

$$\Box\ \frac{A \vdash X}{\Box A \vdash \check{\Box}Y} \qquad \frac{X \vdash \check{\Box}A}{X \vdash \Box A}\ \Box$$

Translation functions

The translations $\tau : \mathcal{L} \to \mathcal{L}_{MT}$ is defined by simultaneous induction as follows:

$$
\begin{array}{rcl}
p^\tau & ::= & p \\
\top^\tau & ::= & \top \\
\bot^\tau & ::= & \bot \\
(A \wedge B)^\tau & ::= & A^\tau \wedge B^\tau \\
(A \vee B)^\tau & ::= & A^\tau \vee B^\tau \\
(\neg A)^\tau & ::= & \Box{\sim}\circ A^\tau
\end{array}
$$

Example

$$\neg\neg A \wedge \neg\neg B \vdash \neg\neg(A \wedge B) \quad \rightsquigarrow \quad \Box{\sim}\circ\Box{\sim}\circ A \wedge \Box{\sim}\circ\Box{\sim}\circ B \vdash \Box{\sim}\circ\Box{\sim}\circ(A \wedge B)$$

Example

$\neg\neg A \wedge \neg\neg B \vdash \neg\neg(A \wedge B) \quad \rightsquigarrow \quad \Box{\sim}\circ\Box{\sim}\circ A \wedge \Box{\sim}\circ\Box{\sim}\circ B \vdash \Box{\sim}\circ\Box{\sim}\circ(A \wedge B)$

- Step 1:

$$
\begin{array}{cc}
 & A \vdash A \\
 & \hline
 & \tilde{\circ}A \vdash \tilde{\circ}A \quad \tilde{\circ} \\
 & \hline
 & \circ A \vdash \tilde{\circ}A \\
 & \hline
 & \tilde{\sim}\tilde{\circ}A \vdash \tilde{\sim}\circ A \quad \text{cont} \\
 & \hline
 & \tilde{\sim}\tilde{\circ}A \vdash {\sim}\circ A \\
 & \hline
 & \tilde{\sim}\tilde{\circ}A \vdash \tilde{\circ}\check{\Box}{\sim}\circ A \quad \tilde{\circ}\check{\Box} \\
 & \hline
 & \tilde{\bullet}\tilde{\sim}\tilde{\circ}A \vdash \check{\Box}{\sim}\circ A \\
 & \hline
 & \tilde{\bullet}\tilde{\sim}\tilde{\circ}A \vdash \Box{\sim}\circ A \\
 & \hline
 & \tilde{\sim}\tilde{\circ}A \vdash \tilde{\circ}\Box{\sim}\circ A \\
 & \hline
 & \tilde{\sim}\tilde{\circ}A \vdash \circ\Box{\sim}\circ A \\
 & \hline
 & \tilde{\sim}\circ\Box{\sim}\circ A \vdash \tilde{\circ}A \\
 & \hline
 & {\sim}\circ\Box{\sim}\circ A \vdash \tilde{\circ}A \\
 & \hline
 & \Box{\sim}\circ\Box{\sim}\circ A \vdash \check{\Box}\tilde{\circ}A \\
\mathrm{W} & \hline
 & \Box{\sim}\circ\Box{\sim}\circ A \hat{\wedge} \Box{\sim}\circ\Box{\sim}\circ B \vdash \check{\Box}\tilde{\circ}A \\
 & \hline
 & \Box{\sim}\circ\Box{\sim}\circ A \wedge \Box{\sim}\circ\Box{\sim}\circ B \vdash \check{\Box}\tilde{\circ}A \\
 & \hline
 & \hat{\blacklozenge}(\Box{\sim}\circ\Box{\sim}\circ A \wedge \Box{\sim}\circ\Box{\sim}\circ B) \vdash \tilde{\circ}A \\
 & \hline
 & \tilde{\bullet}\hat{\blacklozenge}(\Box{\sim}\circ\Box{\sim}\circ A \wedge \Box{\sim}\circ\Box{\sim}\circ B) \vdash A
\end{array}
$$

Example

$\neg\neg A \wedge \neg\neg B \vdash \neg\neg(A \wedge B) \quad \rightsquigarrow \quad \Box{\sim}\circ\Box{\sim}\circ A \wedge \Box{\sim}\circ\Box{\sim}\circ B \vdash \Box{\sim}\circ\Box{\sim}\circ(A \wedge B)$

- Step 1:

$$
\begin{array}{cl}
 & A \vdash A \\
 & \overline{\tilde{\circ} A \vdash \tilde{\circ} A}\ \tilde{\circ} \\
 & \overline{\circ A \vdash \tilde{\circ} A} \\
 & \overline{\tilde{\sim}\tilde{\circ} A \vdash \tilde{\sim}\circ A}\ \text{cont} \\
 & \overline{\tilde{\sim}\tilde{\circ} A \vdash {\sim}\circ A} \\
 & \overline{\tilde{\sim}\tilde{\circ} A \vdash \tilde{\circ}\check{\Box}{\sim}\circ A}\ \tilde{\circ}\check{\Box} \\
 & \overline{\tilde{\bullet}\tilde{\sim}\tilde{\circ} A \vdash \check{\Box}{\sim}\circ A} \\
 & \overline{\tilde{\bullet}\tilde{\sim}\tilde{\circ} A \vdash \Box{\sim}\circ A} \\
 & \overline{\tilde{\sim}\tilde{\circ} A \vdash \tilde{\circ}\Box{\sim}\circ A} \\
 & \overline{\tilde{\sim}\tilde{\circ} A \vdash \circ\Box{\sim}\circ A} \\
 & \overline{\tilde{\sim}\circ\Box{\sim}\circ A \vdash \tilde{\circ} A} \\
 & \overline{{\sim}\circ\Box{\sim}\circ A \vdash \tilde{\circ} A} \\
 & \overline{\Box{\sim}\circ\Box{\sim}\circ A \vdash \check{\Box}\tilde{\circ} A} \\
\mathrm{W} & \overline{\Box{\sim}\circ\Box{\sim}\circ A \mathbin{\hat{\wedge}} \Box{\sim}\circ\Box{\sim}\circ B \vdash \check{\Box}\tilde{\circ} A} \\
 & \overline{\Box{\sim}\circ\Box{\sim}\circ A \wedge \Box{\sim}\circ\Box{\sim}\circ B \vdash \check{\Box}\tilde{\circ} A} \\
 & \overline{\hat{\blacklozenge}(\Box{\sim}\circ\Box{\sim}\circ A \wedge \Box{\sim}\circ\Box{\sim}\circ B) \vdash \tilde{\circ} A} \\
 & \overline{\tilde{\bullet}\hat{\blacklozenge}(\Box{\sim}\circ\Box{\sim}\circ A \wedge \Box{\sim}\circ\Box{\sim}\circ B) \vdash A}
\end{array}
$$

- Step 2: $\tilde{\bullet}\hat{\blacklozenge}(\Box{\sim}\circ\Box{\sim}\circ A \wedge \Box{\sim}\circ\Box{\sim}\circ B) \vdash B$

Example

- Step 3:

$$
\begin{array}{cc}
 & \tilde{\bullet}\hat{\blacklozenge}(\Box{\sim}\circ\Box{\sim}\circ A \wedge \Box{\sim}\circ\Box{\sim}\circ B) \vdash A \qquad \tilde{\bullet}\hat{\blacklozenge}(\Box{\sim}\circ\Box{\sim}\circ A \wedge \Box{\sim}\circ\Box{\sim}\circ B) \vdash B \\
\hline
\mathrm{C} & \tilde{\bullet}\hat{\blacklozenge}(\Box{\sim}\circ\Box{\sim}\circ A \wedge \Box{\sim}\circ\Box{\sim}\circ B) \hat{\wedge} \tilde{\bullet}\hat{\blacklozenge}(\Box{\sim}\circ\Box{\sim}\circ A \wedge \Box{\sim}\circ\Box{\sim}\circ B) \vdash A \wedge B \\
\hline
 & \tilde{\bullet}\hat{\blacklozenge}(\Box{\sim}\circ\Box{\sim}\circ A \wedge \Box{\sim}\circ\Box{\sim}\circ B) \vdash A \wedge B \\
\hline
 & \hat{\blacklozenge}(\Box{\sim}\circ\Box{\sim}\circ A \wedge \Box{\sim}\circ\Box{\sim}\circ B) \vdash \tilde{\circ}(A \wedge B) \\
\hline
 & \hat{\blacklozenge}(\Box{\sim}\circ\Box{\sim}\circ A \wedge \Box{\sim}\circ\Box{\sim}\circ B) \vdash \circ(A \wedge B) \\
\hline
\text{cont} & \tilde{\sim}\circ(A \wedge B) \vdash \tilde{\sim}\hat{\blacklozenge}(\Box{\sim}\circ\Box{\sim}\circ A \wedge \Box{\sim}\circ\Box{\sim}\circ B) \\
\hline
 & {\sim}\circ(A \wedge B) \vdash \tilde{\sim}\hat{\blacklozenge}(\Box{\sim}\circ\Box{\sim}\circ A \wedge \Box{\sim}\circ\Box{\sim}\circ B) \\
\hline
 & \Box{\sim}\circ(A \wedge B) \vdash \check{\Box}\tilde{\sim}\hat{\blacklozenge}(\Box{\sim}\circ\Box{\sim}\circ A \wedge \Box{\sim}\circ\Box{\sim}\circ B) \\
\hline
\tilde{\circ} & \tilde{\circ}\Box{\sim}\circ(A \wedge B) \vdash \tilde{\circ}\check{\Box}\tilde{\sim}\hat{\blacklozenge}(\Box{\sim}\circ\Box{\sim}\circ A \wedge \Box{\sim}\circ\Box{\sim}\circ B) \\
\hline
\tilde{\circ}\check{\Box} & \tilde{\circ}\Box{\sim}\circ(A \wedge B) \vdash \tilde{\sim}\hat{\blacklozenge}(\Box{\sim}\circ\Box{\sim}\circ A \wedge \Box{\sim}\circ\Box{\sim}\circ B) \\
\hline
 & \circ\Box{\sim}\circ(A \wedge B) \vdash \tilde{\sim}\hat{\blacklozenge}(\Box{\sim}\circ\Box{\sim}\circ A \wedge \Box{\sim}\circ\Box{\sim}\circ B) \\
\hline
 & \hat{\blacklozenge}(\Box{\sim}\circ\Box{\sim}\circ A \wedge \Box{\sim}\circ\Box{\sim}\circ B) \vdash \tilde{\sim}\circ\Box{\sim}\circ(A \wedge B) \\
\hline
 & \hat{\blacklozenge}(\Box{\sim}\circ\Box{\sim}\circ A \wedge \Box{\sim}\circ\Box{\sim}\circ B) \vdash {\sim}\circ\Box{\sim}\circ(A \wedge B) \\
\hline
 & \Box{\sim}\circ\Box{\sim}\circ A \wedge \Box{\sim}\circ\Box{\sim}\circ B \vdash \check{\Box}{\sim}\circ\Box{\sim}\circ(A \wedge B) \\
\hline
 & \Box{\sim}\circ\Box{\sim}\circ A \wedge \Box{\sim}\circ\Box{\sim}\circ B \vdash \Box{\sim}\circ\Box{\sim}\circ(A \wedge B)
\end{array}
$$

Equality

Theorem

For all $\mathcal{L}$-formulas A and B and every SM-algebra $\mathbb{L}$,

$$\mathbb{L} \models A \leq B \quad \textit{iff} \quad \mathbb{L}^{+} \models A^{\tau} \leq B^{\tau}.$$

Properties

Theorem (Completeness)

D.SDM is complete with respect to the class of semi-De Morgan algebras.

Theorem (Conservative extension)

D.SDM is a conservative extension of H.SDM.

Theorem (Cut elimination)

If $X \vdash Y$ is derivable in D.SDM, then it is derivable without (Cut).

Theorem (Subformula property)

Any cut-free proof of the sequent $X \vdash Y$ in D.SDM contains only structures over subformulas of formulas in X and Y.

Future work

- extensions to other algebras based on semi-De Morgan algebras, e.g. quasi-De Morgan algebras, demi-p-algebras, weak stone algebras, etc.;

Future work

- extensions to other algebras based on semi-De Morgan algebras, e.g. quasi-De Morgan algebras, demi-p-algebras, weak stone algebras, etc.;
- compatebility frames for semi-De Morgan algebras

Thanks for your attention!