The Cuntz semigroup and the category Cu

Francesc Perera
(joint work with Ramon Antoine and Hannes Thiel
to appear in Mem. Amer. Math. Soc.)
Universitat Autònoma de Barcelona

TOPOLOGY, ALGEBRA, AND CATEGORIES IN LOGIC 2017
Prague, June 2017

Definition:

Let $\mathcal{H}$ be a complex Hilbert space, and let $\mathbb{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$ (with pointwise addition and composition).

C*-algebras

Definition:

Let $\mathcal{H}$ be a complex Hilbert space, and let $\mathbb{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$ (with pointwise addition and composition). A C*-algebra is a *-subalgebra of $\mathbb{B}(\mathcal{H})$, closed under the norm operator topology.

C*-algebras

Definition:

Let $\mathcal{H}$ be a complex Hilbert space, and let $\mathbb{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$ (with pointwise addition and composition). A C*-algebra is a *-subalgebra of $\mathbb{B}(\mathcal{H})$, closed under the norm operator topology.

Abstract characterization:

A C*-algebra is a complex Banach algebra A with an involution $*$ such that $||a^*a|| = ||a||^2$ for all $a \in A$.

C*-algebras

Definition:

Let $\mathcal{H}$ be a complex Hilbert space, and let $\mathbb{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$ (with pointwise addition and composition). A C*-algebra is a *-subalgebra of $\mathbb{B}(\mathcal{H})$, closed under the norm operator topology.

Abstract characterization:

A C*-algebra is a complex Banach algebra A with an involution $*$ such that $||a^*a|| = ||a||^2$ for all $a \in A$.

Examples:

Definition:

Let $\mathcal{H}$ be a complex Hilbert space, and let $\mathbb{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$ (with pointwise addition and composition). A C*-algebra is a *-subalgebra of $\mathbb{B}(\mathcal{H})$, closed under the norm operator topology.

Abstract characterization:

A C*-algebra is a complex Banach algebra A with an involution $*$ such that $\|a^*a\| = \|a\|^2$ for all $a \in A$.

Examples:

(i) $M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_k}(\mathbb{C})$.

C*-algebras

Definition:

Let $\mathcal{H}$ be a complex Hilbert space, and let $\mathbb{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$ (with pointwise addition and composition). A C*-algebra is a *-subalgebra of $\mathbb{B}(\mathcal{H})$, closed under the norm operator topology.

Abstract characterization:

A C*-algebra is a complex Banach algebra A with an involution $*$ such that $||a^*a|| = ||a||^2$ for all $a \in A$.

Examples:

(i) $M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_k}(\mathbb{C})$. (All finite dim'l C*-algebras have this form.)

Definition:

Let $\mathcal{H}$ be a complex Hilbert space, and let $\mathbb{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$ (with pointwise addition and composition). A C*-algebra is a *-subalgebra of $\mathbb{B}(\mathcal{H})$, closed under the norm operator topology.

Abstract characterization:

A C*-algebra is a complex Banach algebra A with an involution $*$ such that $\|a^*a\| = \|a\|^2$ for all $a \in A$.

Examples:

(i) $M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_k}(\mathbb{C})$. (All finite dim'l C*-algebras have this form.)
(ii) For a compact Hausdorff space, $C(X) := \{f\colon X \to \mathbb{C} \mid f \text{cts}\}$, with the sup norm, and $f^*(x) = \overline{f(x)}$,

C*-algebras

Definition:

Let $\mathcal{H}$ be a complex Hilbert space, and let $\mathbb{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$ (with pointwise addition and composition). A C*-algebra is a *-subalgebra of $\mathbb{B}(\mathcal{H})$, closed under the norm operator topology.

Abstract characterization:

A C*-algebra is a complex Banach algebra A with an involution $*$ such that $||a^*a|| = ||a||^2$ for all $a \in A$.

Examples:

(i) $M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_k}(\mathbb{C})$. (All finite dim'l C*-algebras have this form.)
(ii) For a compact Hausdorff space, $C(X) := \{f : X \to \mathbb{C} \mid f \text{cts}\}$, with the sup norm, and $f^*(x) = \overline{f(x)}$, is a commutative example.

Definition:

Let $\mathcal{H}$ be a complex Hilbert space, and let $\mathbb{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$ (with pointwise addition and composition). A C*-algebra is a *-subalgebra of $\mathbb{B}(\mathcal{H})$, closed under the norm operator topology.

Abstract characterization:

A C*-algebra is a complex Banach algebra A with an involution $*$ such that $||a^*a|| = ||a||^2$ for all $a \in A$.

Examples:

(i) $M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_k}(\mathbb{C})$. (All finite dim'l C*-algebras have this form.)
(ii) For a compact Hausdorff space, $C(X) := \{f : X \to \mathbb{C} \mid f \text{ cts}\}$, with the sup norm, and $f^*(x) = \overline{f(x)}$, is a commutative example. (All commutative C*-algebras are of this form.)

Projections, Positive elements, and K-Theory

Projections, Positive elements, and K-Theory

Definition:

A *projection* is an element $p \in A$ such that $p^2 = p = p^*$.

Projections, Positive elements, and K-Theory

Definition:

A *projection* is an element $p \in A$ such that $p^2 = p = p^*$.
A positive element in A is of the form a^*a, for $a \in A$.

Projections, Positive elements, and K-Theory

Definition:

A *projection* is an element $p \in A$ such that $p^2 = p = p^*$.
A positive element in A is of the form a^*a, for $a \in A$. Positive elements generate the C*-algebra linearly.

Projections, Positive elements, and K-Theory

Definition:

A *projection* is an element $p \in A$ such that $p^2 = p = p^*$.
A positive element in A is of the form a^*a, for $a \in A$. Positive elements generate the C*-algebra linearly. Put $A_+ = \{a^*a \mid a \in A\}$.

Projections, Positive elements, and K-Theory

Definition:

A *projection* is an element $p \in A$ such that $p^2 = p = p^*$.
A positive element in A is of the form a^*a, for $a \in A$. Positive elements generate the C*-algebra linearly. Put $A_+ = \{a^*a \mid a \in A\}$.

The group K_0:

For projections p, q, write $p \sim q$ if $p = uu^*$ and $q = u^*u$, some $u \in A$.

Projections, Positive elements, and K-Theory

Definition:

A *projection* is an element $p \in A$ such that $p^2 = p = p^*$.
A positive element in A is of the form a^*a, for $a \in A$. Positive elements generate the C*-algebra linearly. Put $A_+ = \{a^*a \mid a \in A\}$.

The group K_0:

For projections p, q, write $p \sim q$ if $p = uu^*$ and $q = u^*u$, some $u \in A$. This is an equivalence relation, that extends to $M_\infty(A) := \cup M_n(A)$.

Projections, Positive elements, and K-Theory

Definition:

A *projection* is an element $p \in A$ such that $p^2 = p = p^*$.
A positive element in A is of the form a^*a, for $a \in A$. Positive elements generate the C*-algebra linearly. Put $A_+ = \{a^*a \mid a \in A\}$.

The group K_0:

For projections p, q, write $p \sim q$ if $p = uu^*$ and $q = u^*u$, some $u \in A$. This is an equivalence relation, that extends to $M_\infty(A) := \cup M_n(A)$. Let $V(A) = \mathrm{Proj}(M_\infty(A))/\sim$.

Projections, Positive elements, and K-Theory

Definition:

A *projection* is an element $p \in A$ such that $p^2 = p = p^*$.
A positive element in A is of the form a^*a, for $a \in A$. Positive elements generate the C*-algebra linearly. Put $A_+ = \{a^*a \mid a \in A\}$.

The group K_0:

For projections p, q, write $p \sim q$ if $p = uu^*$ and $q = u^*u$, some $u \in A$. This is an equivalence relation, that extends to $M_\infty(A) := \cup M_n(A)$. Let $V(A) = \mathrm{Proj}(M_\infty(A))/\sim$. This is an abelian semigroup, with operation $[p] + [q] := [\left(\begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix}\right)]$.
Define $K_0(A) = G(V(A))$, the Grothendieck group of $V(A)$.

The Elliott Invariant:

Let $\mathrm{Ell}(A) = ((K_0(A), K_0(A)^+, [1_A]), K_1(A), T(A), r_A)$, for any C*-algebra A.

Elliott's Classification Program

The Elliott Invariant:

Let $\mathrm{Ell}(A) = ((K_0(A), K_0(A)^+, [1_A]), K_1(A), T(A), r_A)$, for any C*-algebra A.

Elliott's Classification Conjecture:

For simple, unital C*-algebras A and B in a reasonably large class:

Elliott's Classification Program

The Elliott Invariant:

Let $\mathrm{Ell}(A) = ((K_0(A), K_0(A)^+, [1_A]), K_1(A), T(A), r_A)$, for any C*-algebra A.

Elliott's Classification Conjecture:

For simple, unital C*-algebras A and B in a reasonably large class:

$$\mathrm{Ell}(A) \cong \mathrm{Ell}(B) \iff A \cong B.$$

Elliott's Classification Program

The Elliott Invariant:

Let $\mathrm{Ell}(A) = ((K_0(A), K_0(A)^+, [1_A]), K_1(A), T(A), r_A)$, for any C*-algebra A.

Elliott's Classification Conjecture:

For simple, unital C*-algebras A and B in a reasonably large class:

$$\mathrm{Ell}(A) \cong \mathrm{Ell}(B) \iff A \cong B.$$

Progress to resolve this conjecture has been tremendous over the last decades, until its near completion.

Elliott's Classification Program

The Elliott Invariant:

Let $\mathrm{Ell}(A) = ((K_0(A), K_0(A)^+, [1_A]), K_1(A), T(A), r_A)$, for any C*-algebra A.

Elliott's Classification Conjecture:

For simple, unital C*-algebras A and B in a reasonably large class:

$$\mathrm{Ell}(A) \cong \mathrm{Ell}(B) \iff A \cong B.$$

Progress to resolve this conjecture has been tremendous over the last decades, until its near completion. Roughly, this class consists of those *simple* unital C*-algebras A

The Elliott Invariant:

Let $\mathrm{Ell}(A) = ((K_0(A), K_0(A)^+, [1_A]), K_1(A), T(A), r_A)$, for any C*-algebra A.

Elliott's Classification Conjecture:

For simple, unital C*-algebras A and B in a reasonably large class:

$$\mathrm{Ell}(A) \cong \mathrm{Ell}(B) \iff A \cong B.$$

Progress to resolve this conjecture has been tremendous over the last decades, until its near completion. Roughly, this class consists of those *simple* unital C*-algebras A such that
$A \cong A \otimes \mathcal{Z}$

The Elliott Invariant:

Let $\mathrm{Ell}(A) = ((K_0(A), K_0(A)^+, [1_A]), K_1(A), T(A), r_A)$, for any C*-algebra A.

Elliott's Classification Conjecture:

For simple, unital C*-algebras A and B in a reasonably large class:

$$\mathrm{Ell}(A) \cong \mathrm{Ell}(B) \iff A \cong B.$$

Progress to resolve this conjecture has been tremendous over the last decades, until its near completion. Roughly, this class consists of those *simple* unital C*-algebras A such that
$A \cong A \otimes \mathcal{Z}$
where $\mathcal{Z}$ is the so called Jiang-Su algebra (an infinite dimensional C*-algebra with K-Theory as the complex numbers).

The Elliott Invariant:

Let $\mathrm{Ell}(A) = ((K_0(A), K_0(A)^+, [1_A]), K_1(A), T(A), r_A)$, for any C*-algebra A.

Elliott's Classification Conjecture:

For simple, unital C*-algebras A and B in a reasonably large class:

$$\mathrm{Ell}(A) \cong \mathrm{Ell}(B) \iff A \cong B.$$

Progress to resolve this conjecture has been tremendous over the last decades, until its near completion. Roughly, this class consists of those *simple* unital C*-algebras A such that
$A \cong A \otimes \mathcal{Z}$
where $\mathcal{Z}$ is the so called Jiang-Su algebra (an infinite dimensional C*-algebra with K-Theory as the complex numbers). Some progress exists in the non-simple setting (and this is a direction to explore).

The Cuntz semigroup - classical definition

Definition (Cuntz, 1978):

Let A be a C*-algebra. If $a, b \in A_+$, write $a \precsim b$ if

Definition (Cuntz, 1978):

Let A be a C*-algebra. If $a, b \in A_+$, write $a \precsim b$ if

$$a = \lim_n x_n b x_n^* \text{ for a sequence } x_n \in A\,.$$

(We say that a is Cuntz subequivalent to b.)

The Cuntz semigroup - classical definition

Definition (Cuntz, 1978):

Let A be a C*-algebra. If $a, b \in A_+$, write $a \precsim b$ if

$$a = \lim_n x_n b x_n^* \text{ for a sequence } x_n \in A\,.$$

(We say that a is Cuntz subequivalent to b.)
Write $a \sim b$ if both $a \precsim b$ and $b \precsim a$ occur.
(We say that a and b are Cuntz equivalent.)

The Cuntz semigroup - classical definition

Definition (Cuntz, 1978):

Let A be a C*-algebra. If $a, b \in A_+$, write $a \precsim b$ if

$$a = \lim_n x_n b x_n^* \text{ for a sequence } x_n \in A\,.$$

(We say that a is Cuntz subequivalent to b.)
Write $a \sim b$ if both $a \precsim b$ and $b \precsim a$ occur.
(We say that a and b are Cuntz equivalent.)
The Cuntz semigroup of A is

$$W(A) = M_\infty(A)_+/\!\sim$$

The Cuntz semigroup - classical definition

Definition (Cuntz, 1978):

Let A be a C*-algebra. If $a, b \in A_+$, write $a \precsim b$ if

$$a = \lim_n x_n b x_n^* \text{ for a sequence } x_n \in A\,.$$

(We say that a is Cuntz subequivalent to b.)
Write $a \sim b$ if both $a \precsim b$ and $b \precsim a$ occur.
(We say that a and b are Cuntz equivalent.)
The Cuntz semigroup of A is

$$W(A) = M_\infty(A)_+/\sim$$

with addition $[a] + [b] = [\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)]$ and order $[a] \leq [b]$ if $a \precsim b$.

The Cuntz semigroup - classical definition

Definition (Cuntz, 1978):

Let A be a C*-algebra. If $a, b \in A_+$, write $a \precsim b$ if

$$a = \lim_n x_n b x_n^* \text{ for a sequence } x_n \in A\,.$$

(We say that a is Cuntz subequivalent to b.)
Write $a \sim b$ if both $a \precsim b$ and $b \precsim a$ occur.
(We say that a and b are Cuntz equivalent.)
The Cuntz semigroup of A is

$$W(A) = M_\infty(A)_+/\!\sim$$

with addition $[a] + [b] = [\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)]$ and order $[a] \leq [b]$ if $a \precsim b$. There is a map $V(A) \to W(A)$ sending $[p]$ to $[p]$. It is injective if A is stably finite.

The Cuntz semigroup - classical definition

Definition (Cuntz, 1978):

Let A be a C*-algebra. If $a, b \in A_+$, write $a \precsim b$ if

$$a = \lim_n x_n b x_n^* \text{ for a sequence } x_n \in A\,.$$

(We say that a is Cuntz subequivalent to b.)
Write $a \sim b$ if both $a \precsim b$ and $b \precsim a$ occur.
(We say that a and b are Cuntz equivalent.)
The Cuntz semigroup of A is

$$W(A) = M_\infty(A)_+/\sim$$

with addition $[a] + [b] = [\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)]$ and order $[a] \leq [b]$ if $a \precsim b$. There is a map $V(A) \to W(A)$ sending $[p]$ to $[p]$. It is injective if A is stably finite. The semigroup $W(A)$ is an example of a *predomain*, as defined by Keimel.

The main drawback of the classical definition: $W(-)$ is not a continuous functor from the category of C*-algebras to the category of semigroups.

The Cuntz semigroup - current definition

The main drawback of the classical definition: $W(-)$ is not a continuous functor from the category of C*-algebras to the category of semigroups.

Definition (Coward, Elliott, Ivanescu, 2008):

The (completed) Cuntz semigroup of a C*-algebra A is

$$\mathrm{Cu}(A) = (A \otimes \mathcal{K})_+/\sim \; (= W(A \otimes \mathcal{K}))$$

The Cuntz semigroup - current definition

The main drawback of the classical definition: $W(-)$ is not a continuous functor from the category of C*-algebras to the category of semigroups.

Definition (Coward, Elliott, Ivanescu, 2008):

The (completed) Cuntz semigroup of a C*-algebra A is

$$\mathrm{Cu}(A) = (A \otimes \mathcal{K})_+/\sim \; (= W(A \otimes \mathcal{K}))$$

With this definition, the assignment $A \mapsto \mathrm{Cu}(A)$ is a sequentially continuous functor.

The Cuntz semigroup - current definition

The main drawback of the classical definition: $W(-)$ is not a continuous functor from the category of C*-algebras to the category of semigroups.

Definition (Coward, Elliott, Ivanescu, 2008):

The (completed) Cuntz semigroup of a C*-algebra A is

$$\mathrm{Cu}(A) = (A \otimes \mathcal{K})_+/\sim \; (= W(A \otimes \mathcal{K}))$$

With this definition, the assignment $A \mapsto \mathrm{Cu}(A)$ is a sequentially continuous functor.
But, between which categories?

The Category Cu

Theorem/Definition: (**Coward, Elliott, Ivanescu 2008**)

Cu is a category of ordered commutative semigroups S with 0 such that

(O1) Every increasing sequence has in S a supremum.

Theorem/Definition: (Coward, Elliott, Ivanescu 2008)

Cu is a category of ordered commutative semigroups S with 0 such that

(O1) Every increasing sequence has in S a supremum.

(O2) Any element $a \in S$ can be written as $a = \sup a_n$, where $a_n \ll a_{n+1}$.

(O3) The relation $\ll$ is compatible with addition. $\ll$

Theorem/Definition: (Coward, Elliott, Ivanescu 2008)

Cu is a category of ordered commutative semigroups S with 0 such that

(O1) Every increasing sequence has in S a supremum.

(O2) Any element $a \in S$ can be written as $a = \sup a_n$, where $a_n \ll a_{n+1}$.

(O3) The relation $\ll$ is compatible with addition. $\ll$

(O4) Addition and suprema are also compatible.

Theorem/Definition: (Coward, Elliott, Ivanescu 2008)

Cu is a category of ordered commutative semigroups S with 0 such that

(O1) Every increasing sequence has in S a supremum.

(O2) Any element $a \in S$ can be written as $a = \sup a_n$, where $a_n \ll a_{n+1}$.

(O3) The relation $\ll$ is compatible with addition. $\ll$

(O4) Addition and suprema are also compatible.

Maps are semigroup morphisms that preserve zero, order, suprema, and the relation $\ll$.

Theorem/Definition: (**Coward, Elliott, Ivanescu 2008**)

Cu is a category of ordered commutative semigroups S with 0 such that

(O1) Every increasing sequence has in S a supremum.

(O2) Any element $a \in S$ can be written as $a = \sup a_n$, where $a_n \ll a_{n+1}$.

(O3) The relation $\ll$ is compatible with addition. $\ll$

(O4) Addition and suprema are also compatible.

Maps are semigroup morphisms that preserve zero, order, suprema, and the relation $\ll$.

The category Cu has countable inductive limits and $\mathrm{Cu}(-) = W(- \otimes \mathbb{K})$ defines a sequentially continuous functor from the category of C*-algebras to Cu.

Some examples

Some examples

The following semigroups belong to Cu.

Some examples

The following semigroups belong to Cu.

- $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

Some examples

The following semigroups belong to Cu.

- $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. (Appears as $\mathrm{Cu}(\mathbb{C})$.)

Some examples

The following semigroups belong to Cu.

- $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. (Appears as $\mathrm{Cu}(\mathbb{C})$.)
- $\mathrm{Lsc}(X, \overline{\mathbb{N}})$.

Some examples

The following semigroups belong to Cu.

- $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. (Appears as $\mathrm{Cu}(\mathbb{C})$.)
- $\mathrm{Lsc}(X, \overline{\mathbb{N}})$. More generally, $\mathrm{Lsc}(X, S)$ for $S \in \mathrm{Cu}$.

Some examples

The following semigroups belong to Cu.

- $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. (Appears as $\mathrm{Cu}(\mathbb{C})$.)
- $\mathrm{Lsc}(X, \overline{\mathbb{N}})$. More generally, $\mathrm{Lsc}(X, S)$ for $S \in \mathrm{Cu}$. It belongs to Cu if X is finite dim'l.

Some examples

The following semigroups belong to Cu.

- $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. (Appears as $\mathrm{Cu}(\mathbb{C})$.)
- $\mathrm{Lsc}(X, \overline{\mathbb{N}})$. More generally, $\mathrm{Lsc}(X, S)$ for $S \in \mathrm{Cu}$. It belongs to Cu if X is finite dim'l. (If A is a C*-algebra with enough trivial K_1, then $\mathrm{Cu}(C(X, A)) \cong \mathrm{Lsc}(X, \mathrm{Cu}(A))$ if X is one-dimensional.)

Some examples

The following semigroups belong to Cu.

- $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. (Appears as $\mathrm{Cu}(\mathbb{C})$.)
- $\mathrm{Lsc}(X, \overline{\mathbb{N}})$. More generally, $\mathrm{Lsc}(X, S)$ for $S \in \mathrm{Cu}$. It belongs to Cu if X is finite dim'l. (If A is a C*-algebra with enough trivial K_1, then $\mathrm{Cu}(C(X, A)) \cong \mathrm{Lsc}(X, \mathrm{Cu}(A))$ if X is one-dimensional.)
- $Z = \mathbb{N} \sqcup (0, \infty]$.

Some examples

The following semigroups belong to Cu.

- $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. (Appears as $\mathrm{Cu}(\mathbb{C})$.)
- $\mathrm{Lsc}(X, \overline{\mathbb{N}})$. More generally, $\mathrm{Lsc}(X, S)$ for $S \in \mathrm{Cu}$. It belongs to Cu if X is finite dim'l. (If A is a C*-algebra with enough trivial K_1, then $\mathrm{Cu}(C(X, A)) \cong \mathrm{Lsc}(X, \mathrm{Cu}(A))$ if X is one-dimensional.)
- $Z = \mathbb{N} \sqcup (0, \infty]$. (It appears as $\mathrm{Cu}(\mathcal{Z})$.) Z

Some examples

The following semigroups belong to Cu.

- $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. (Appears as $\mathrm{Cu}(\mathbb{C})$.)
- $\mathrm{Lsc}(X, \overline{\mathbb{N}})$. More generally, $\mathrm{Lsc}(X, S)$ for $S \in \mathrm{Cu}$. It belongs to Cu if X is finite dim'l. (If A is a C*-algebra with enough trivial K_1, then $\mathrm{Cu}(C(X, A)) \cong \mathrm{Lsc}(X, \mathrm{Cu}(A))$ if X is one-dimensional.)
- $Z = \mathbb{N} \sqcup (0, \infty]$. (It appears as $\mathrm{Cu}(\mathcal{Z})$.) Z
- For a supernatural number $q \neq 1$ such that $q^2 = q$, put $R_q = \mathbb{N}[\frac{1}{q}] \sqcup (0, \infty]$.

Some examples

The following semigroups belong to Cu.

- $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. (Appears as $\mathrm{Cu}(\mathbb{C})$.)
- $\mathrm{Lsc}(X, \overline{\mathbb{N}})$. More generally, $\mathrm{Lsc}(X, S)$ for $S \in \mathrm{Cu}$. It belongs to Cu if X is finite dim'l. (If A is a C*-algebra with enough trivial K_1, then $\mathrm{Cu}(C(X, A)) \cong \mathrm{Lsc}(X, \mathrm{Cu}(A))$ if X is one-dimensional.)
- $Z = \mathbb{N} \sqcup (0, \infty]$. (It appears as $\mathrm{Cu}(\mathcal{Z})$.) Z
- For a supernatural number $q \neq 1$ such that $q^2 = q$, put $R_q = \mathbb{N}[\frac{1}{q}] \sqcup (0, \infty]$. (It appears as $\mathrm{Cu}(M_q)$, the so called UHF-algebra of type q.)

Some examples

The following semigroups belong to Cu.

- $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. (Appears as $\mathrm{Cu}(\mathbb{C})$.)
- $\mathrm{Lsc}(X, \overline{\mathbb{N}})$. More generally, $\mathrm{Lsc}(X, S)$ for $S \in \mathrm{Cu}$. It belongs to Cu if X is finite dim'l. (If A is a C*-algebra with enough trivial K_1, then $\mathrm{Cu}(C(X, A)) \cong \mathrm{Lsc}(X, \mathrm{Cu}(A))$ if X is one-dimensional.)
- $Z = \mathbb{N} \sqcup (0, \infty]$. (It appears as $\mathrm{Cu}(\mathcal{Z})$.) ▸ Z
- For a supernatural number $q \neq 1$ such that $q^2 = q$, put $R_q = \mathbb{N}[\frac{1}{q}] \sqcup (0, \infty]$. (It appears as $\mathrm{Cu}(M_q)$, the so called UHF-algebra of type q.)
- Let $Z' = \{0, 1, 1', 2, 3, 4, \dots\} \sqcup (0, \infty]$, with addition as in Z except that $k + 1' = k + 1$.

Some examples

The following semigroups belong to Cu.

- $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. (Appears as $\mathrm{Cu}(\mathbb{C})$.)
- $\mathrm{Lsc}(X, \overline{\mathbb{N}})$. More generally, $\mathrm{Lsc}(X, S)$ for $S \in \mathrm{Cu}$. It belongs to Cu if X is finite dim'l. (If A is a C*-algebra with enough trivial K_1, then $\mathrm{Cu}(C(X, A)) \cong \mathrm{Lsc}(X, \mathrm{Cu}(A))$ if X is one-dimensional.)
- $Z = \mathbb{N} \sqcup (0, \infty]$. (It appears as $\mathrm{Cu}(\mathcal{Z})$.) Z
- For a supernatural number $q \neq 1$ such that $q^2 = q$, put $R_q = \mathbb{N}[\frac{1}{q}] \sqcup (0, \infty]$. (It appears as $\mathrm{Cu}(M_q)$, the so called UHF-algebra of type q.)
- Let $Z' = \{0, 1, 1', 2, 3, 4, \dots\} \sqcup (0, \infty]$, with addition as in Z except that $k + 1' = k + 1$. It is not known if there is a C*-algebra A with $\mathrm{Cu}(A) \cong Z'$.

Let C*=category of C*-algebras and *-homomorphisms, and C$^*_{\mathrm{loc}}$=category of local C*-algebras.

Relationship between $\mathrm{W}(A)$ and $\mathrm{Cu}(A)$

Let C*=category of C*-algebras and *-homomorphisms, and C$^*_{\mathrm{loc}}$=category of local C*-algebras.

Theorem (Antoine, P, Thiel):

(i) There is a category W of ordered semigroups with 0 that admits arbitrary inductive limits, such that $\mathrm{W}(A)$ ∈W, and such that $A \mapsto \mathrm{W}(A)$ is a continuous functor. W

Relationship between $\mathrm{W}(A)$ and $\mathrm{Cu}(A)$

Let C*=category of C*-algebras and *-homomorphisms, and C$^*_{\mathrm{loc}}$=category of local C*-algebras.

Theorem (Antoine, P, Thiel):

(i) There is a category W of ordered semigroups with 0 that admits arbitrary inductive limits, such that $\mathrm{W}(A)$ ∈W, and such that $A \mapsto \mathrm{W}(A)$ is a continuous functor. ▸ W

(ii) $\mathrm{Cu} \subset$ W as a full reflective subcategory.

Relationship between $\mathrm{W}(A)$ and $\mathrm{Cu}(A)$

Let C^*=category of C^*-algebras and *-homomorphisms, and $\mathrm{C}^*_{\mathrm{loc}}$=category of local C^*-algebras.

Theorem (Antoine, P, Thiel):

(i) There is a category W of ordered semigroups with 0 that admits arbitrary inductive limits, such that $\mathrm{W}(A) \in$W, and such that $A \mapsto \mathrm{W}(A)$ is a continuous functor. ▸ W

(ii) $\mathrm{Cu} \subset$ W as a full reflective subcategory.

(iii) There is a commutative diagram (up to natural isomorphisms):

$$\begin{tikzcd} \mathrm{C}^*_{\mathrm{loc}} \arrow[r, "\mathrm{W}"] \arrow[d, bend right, "\gamma"'] & \mathrm{W} \arrow[d, bend left, "\gamma"] \\ \mathrm{C}^* \arrow[r, "\mathrm{Cu}"] \arrow[u, hook] & \mathrm{Cu} \arrow[u, hook] \end{tikzcd}$$

Relationship between $\mathrm{W}(A)$ and $\mathrm{Cu}(A)$

Let C*=category of C*-algebras and *-homomorphisms, and C$^*_{\mathrm{loc}}$=category of local C*-algebras.

Theorem (Antoine, P, Thiel):

(i) There is a category W of ordered semigroups with 0 that admits arbitrary inductive limits, such that $\mathrm{W}(A)$ $\in$W, and such that $A \mapsto \mathrm{W}(A)$ is a continuous functor. W

(ii) $\mathrm{Cu} \subset$ W as a full reflective subcategory.

(iii) There is a commutative diagram (up to natural isomorphisms):

$$\begin{tikzcd} \mathrm{C}^*_{\mathrm{loc}} \arrow[r, "\mathrm{W}"] \arrow[d, bend right, "\gamma"'] & \mathrm{W} \arrow[d, bend left, "\gamma"] \\ \mathrm{C}^* \arrow[r, "\mathrm{Cu}"] \arrow[u, hook] & \mathrm{Cu} \arrow[u, hook] \end{tikzcd}$$

($\Longrightarrow$ $A \mapsto \mathrm{Cu}(A)$ is arbitrarily continuous)

Relationship between $\mathrm{Cu}(A)$ and the Elliott invariant

Recall that the Elliott invariant is
$\mathrm{Ell}(A) = ((K_0(A), K_0(A)^+, [1_A]), K_1(A), T(A), r_A)$.

Relationship between $\mathrm{Cu}(A)$ and the Elliott invariant

Recall that the Elliott invariant is
$\mathrm{Ell}(A) = ((K_0(A), K_0(A)^+, [1_A]), K_1(A), T(A), r_A)$. It is natural to ask what is the relationship between $\mathrm{Cu}(A)$ and $\mathrm{Ell}(A)$. This can be derived from:

Relationship between $\mathrm{Cu}(A)$ and the Elliott invariant

Recall that the Elliott invariant is
$\mathrm{Ell}(A) = ((K_0(A), K_0(A)^+, [1_A]), K_1(A), T(A), r_A)$. It is natural to ask what is the relationship between $\mathrm{Cu}(A)$ and $\mathrm{Ell}(A)$. This can be derived from:

Theorem (Brown, P, Toms, 2008):

Let A be a simple, separable, nuclear, $\mathcal{Z}$-stable C$*$-algebra. Then

$$\mathrm{Cu}(A) \cong V(A) \sqcup \mathrm{LAff}(T(A))^{++}$$

Relationship between $\mathrm{Cu}(A)$ and the Elliott invariant

Recall that the Elliott invariant is
$\mathrm{Ell}(A) = ((K_0(A), K_0(A)^+, [1_A]), K_1(A), T(A), r_A)$. It is natural to ask what is the relationship between $\mathrm{Cu}(A)$ and $\mathrm{Ell}(A)$. This can be derived from:

Theorem (Brown, P, Toms, 2008):

Let A be a simple, separable, nuclear, $\mathcal{Z}$-stable C$*$-algebra. Then

$$\mathrm{Cu}(A) \cong V(A) \sqcup \mathrm{LAff}(T(A))^{++}$$

Theorem (Tikuisis, 2011; Antoine, Dadarlat, P, Santiago, 2014):

Let A be a simple, separable, nuclear, finite, $\mathcal{Z}$-stable C$*$-algebra. Then

$$\mathrm{Cu}(C(\mathbb{T}, A)) \cong (\{0\} \sqcup (V(A)^* \times K_1(A))) \sqcup \mathrm{Lsc}_{\mathrm{nc}}(\mathbb{T}, \mathrm{Cu}(A))$$

Relationship between $\mathrm{Cu}(A)$ and the Elliott invariant

Recall that the Elliott invariant is
$\mathrm{Ell}(A) = ((K_0(A), K_0(A)^+, [1_A]), K_1(A), T(A), r_A)$. It is natural to ask what is the relationship between $\mathrm{Cu}(A)$ and $\mathrm{Ell}(A)$. This can be derived from:

Theorem (Brown, P, Toms, 2008):

Let A be a simple, separable, nuclear, $\mathcal{Z}$-stable C$*$-algebra. Then

$$\mathrm{Cu}(A) \cong V(A) \sqcup \mathrm{LAff}(T(A))^{++}$$

Theorem (Tikuisis, 2011; Antoine, Dadarlat, P, Santiago, 2014):

Let A be a simple, separable, nuclear, finite, $\mathcal{Z}$-stable C$*$-algebra. Then

$$\mathrm{Cu}(C(\mathbb{T}, A)) \cong (\{0\} \sqcup (V(A)^* \times K_1(A))) \sqcup \mathrm{Lsc}_{\mathrm{nc}}(\mathbb{T}, \mathrm{Cu}(A))$$

It follows from these results that $\mathrm{Cu}(C(\mathbb{T}, -))$ and $\mathrm{Ell}(-)$ determine one another functorially.

Structure of the category Cu

Theorem (Antoine, P, Thiel):

The category Cu is a symmetric, monoidal, closed category, that is complete and cocomplete.

Theorem (Antoine, P, Thiel):

The category Cu is a symmetric, monoidal, closed category, that is complete and cocomplete.

Thus, there is a bifunctor $-\otimes_{\mathrm{Cu}}-\colon \mathrm{Cu}\times\mathrm{Cu}\to\mathrm{Cu}$ with $\overline{\mathbb{N}}$ as identity object. This bifunctor induces associative and commutative (natural) isomorphisms.

Structure of the category Cu

Theorem (Antoine, P, Thiel):

The category Cu is a symmetric, monoidal, closed category, that is complete and cocomplete.

Thus, there is a bifunctor $-\otimes_{\mathrm{Cu}} -\colon\ \mathrm{Cu}\times\mathrm{Cu}\to\mathrm{Cu}$ with $\overline{\mathbb{N}}$ as identity object. This bifunctor induces associative and commutative (natural) isomorphisms. Idea of construction: For $S,T\in\mathrm{Cu}$, construct $S\otimes T$ as partially ordered semigroups; define an auxiliary relation

$$x\prec\sum_i y_i\otimes z_i \iff \exists y_i'\ll y_i, z_i'\ll z_i \text{ with } x\leq\sum_i y_i'\otimes z_i'$$

Theorem (Antoine, P, Thiel):

The category Cu is a symmetric, monoidal, closed category, that is complete and cocomplete.

Thus, there is a bifunctor $-\otimes_{\mathrm{Cu}} -\colon \mathrm{Cu}\times\mathrm{Cu}\to\mathrm{Cu}$ with $\overline{\mathbb{N}}$ as identity object. This bifunctor induces associative and commutative (natural) isomorphisms. Idea of construction: For $S, T\in\mathrm{Cu}$, construct $S\otimes T$ as partially ordered semigroups; define an auxiliary relation

$$x\prec\sum_i y_i\otimes z_i \iff \exists y_i'\ll y_i, z_i'\ll z_i \text{ with } x\leq\sum_i y_i'\otimes z_i'$$

Then $(S\otimes T,\prec)$ is a W-semigroup,

Theorem (Antoine, P, Thiel):

The category Cu is a symmetric, monoidal, closed category, that is complete and cocomplete.

Thus, there is a bifunctor $-\otimes_{\mathrm{Cu}} -\colon \mathrm{Cu}\times\mathrm{Cu}\to\mathrm{Cu}$ with $\overline{\mathbb{N}}$ as identity object. This bifunctor induces associative and commutative (natural) isomorphisms. Idea of construction: For $S, T\in\mathrm{Cu}$, construct $S\otimes T$ as partially ordered semigroups; define an auxiliary relation

$$x\prec\sum_i y_i\otimes z_i \iff \exists y_i'\ll y_i, z_i'\ll z_i \text{ with } x\leq\sum_i y_i'\otimes z_i'$$

Then $(S\otimes T,\prec)$ is a W-semigroup, and one takes

$$S\otimes_{\mathrm{Cu}} T=\gamma(S\otimes T,\prec)\,.$$

Theorem (Antoine, P, Thiel):

The category Cu is a symmetric, monoidal, closed category, that is complete and cocomplete.

Thus, there is a bifunctor $-\otimes_{\mathrm{Cu}} -\colon \mathrm{Cu}\times\mathrm{Cu}\to\mathrm{Cu}$ with $\overline{\mathbb{N}}$ as identity object. This bifunctor induces associative and commutative (natural) isomorphisms. Idea of construction: For $S, T\in\mathrm{Cu}$, construct $S\otimes T$ as partially ordered semigroups; define an auxiliary relation

$$x \prec \sum_i y_i\otimes z_i \iff \exists y_i'\ll y_i, z_i'\ll z_i \text{ with } x\leq \sum_i y_i'\otimes z_i'$$

Then $(S\otimes T,\prec)$ is a W-semigroup, and one takes

$$S\otimes_{\mathrm{Cu}} T=\gamma(S\otimes T,\prec)\,.$$

$S\otimes_{\mathrm{Cu}} T$ represents the bimorphism functor $\mathrm{BiCu}(S\times T,-)$.

Thank You!!

Compact Containment

Definition:

Given $a, b \in S$,

$a \ll b \iff (b \leq \sup c_n$ with (c_n) increasing $\implies a \leq c_{n_0}$ for some $n_0 \geq 0)$.

A sequence is (a_n) is **rapidly increasing** if $a_n \ll a_{n+1}$ for all n.

Compact Containment

Definition:

Given $a, b \in S$,

$$a \ll b \iff (b \leq \sup c_n \text{ with } (c_n) \text{ increasing } \implies a \leq c_{n_0} \text{ for some } n_0 \geq 0).$$

A sequence is (a_n) is **rapidly increasing** if $a_n \ll a_{n+1}$ for all n.

An element $a \in X$ is termed **compact** if $a \ll a$.

Compact Containment

Definition:

Given $a, b \in S$,

$a \ll b \iff (b \leq \sup c_n$ with (c_n) increasing $\implies a \leq c_{n_0}$ for some $n_0 \geq 0)$.

A sequence is (a_n) is **rapidly increasing** if $a_n \ll a_{n+1}$ for all n.

An element $a \in X$ is termed **compact** if $a \ll a$.

If A is a C*-algebra, then $[(a-\epsilon)_+] \ll [a]\ \forall \epsilon > 0$.

Compact Containment

Definition:

Given $a, b \in S$,

$a \ll b \iff (b \leq \sup c_n$ with (c_n) increasing $\implies a \leq c_{n_0}$ for some $n_0 \geq 0)$.

A sequence is (a_n) is **rapidly increasing** if $a_n \ll a_{n+1}$ for all n.

An element $a \in X$ is termed **compact** if $a \ll a$.

If A is a C*-algebra, then $[(a-\epsilon)_+] \ll [a]$ $\forall \epsilon > 0$.
And $[a] = \sup[(a-1/n)_+]$.

Compact Containment

Definition:

Given $a, b \in S$,

$a \ll b \iff (b \leq \sup c_n$ with (c_n) increasing $\implies a \leq c_{n_0}$ for some $n_0 \geq 0)$.

A sequence is (a_n) is **rapidly increasing** if $a_n \ll a_{n+1}$ for all n.

An element $a \in X$ is termed **compact** if $a \ll a$.

If A is a C*-algebra, then $[(a-\epsilon)_+] \ll [a]\ \forall \epsilon > 0$.
And $[a] = \sup[(a-1/n)_+]$.
If A is stably finite, the compact elements are precisely $[p]$, where p is a projection.

The Category W

Definition:

Let S be a positively ordered semigroup. An **auxiliary relation** on S is a binary relation $\prec$ such that

The Category W

Definition:

Let S be a positively ordered semigroup. An **auxiliary relation** on S is a binary relation $\prec$ such that

(i) $a \prec b \implies a \leq b$

Definition:

Let S be a positively ordered semigroup. An **auxiliary relation** on S is a binary relation $\prec$ such that

(i) $a \prec b \implies a \leq b$

(ii) $a \leq b \prec c \leq d \implies a \prec d$

Definition:

Let S be a positively ordered semigroup. An **auxiliary relation** on S is a binary relation $\prec$ such that

(i) $a \prec b \implies a \leq b$

(ii) $a \leq b \prec c \leq d \implies a \prec d$

(iii) $0 \prec a$

The Category W

Definition:

Let S be a positively ordered semigroup. An **auxiliary relation** on S is a binary relation $\prec$ such that

(i) $a \prec b \implies a \leq b$

(ii) $a \leq b \prec c \leq d \implies a \prec d$

(iii) $0 \prec a$

Examples:

1. If $S \in \mathrm{Cu}$, then $\prec = \ll$ is an auxiliary relation.
2. Let A be a local C*-algebra. For $[a], [b] \in \mathrm{W}(A)$, set $[a] \prec [b]$ if $[a] \leq [(b-\epsilon)_+]$. Then $\prec$ is an auxiliary relation on $\mathrm{W}(A)$.

The Category W cont'd

Definition: (Antoine, P, Thiel)

The category W has as objects $(S, \prec)$, where S is positively ordered, $\prec$ is an auxiliary relation, and

W1: For each $a \in S$, the set $a^{\prec}$ has a $\prec$-increasing countable cofinal subset.

The Category W cont'd

Definition: (Antoine, P, Thiel)

The category W has as objects $(S, \prec)$, where S is positively ordered, $\prec$ is an auxiliary relation, and

W1: For each $a \in S$, the set $a^{\prec}$ has a $\prec$-increasing countable cofinal subset.

W2: $a = \sup a^{\prec}$.

The Category W cont'd

Definition: (Antoine, P, Thiel)

The category W has as objects $(S, \prec)$, where S is positively ordered, $\prec$ is an auxiliary relation, and

W1: For each $a \in S$, the set $a^{\prec}$ has a $\prec$-increasing countable cofinal subset.

W2: $a = \sup a^{\prec}$.

W3: $\prec$ and $+$ are compatible.

W4: $a \prec b + c \implies a \prec b' + c'$ with $b' \prec b$, $c' \prec c$.

Theorem (Jiang, Su, 2000):

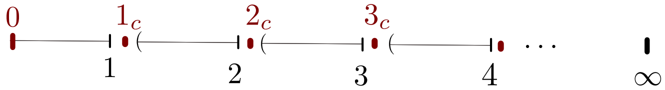
There exists a simple, unital, projectionless and infinite dimensional C*-algebra $\mathcal{Z}$, which is the inductive limit of dimension drop algebras, has a unique trace, and the same K-theory as $\mathbb{C}$.

The Jiang-Su Algebra $\mathcal{Z}$

Theorem (Jiang, Su, 2000):

There exists a simple, unital, projectionless and infinite dimensional C*-algebra $\mathcal{Z}$, which is the inductive limit of dimension drop algebras, has a unique trace, and the same K-theory as $\mathbb{C}$.

$\mathrm{Cu}(\mathcal{Z}) = (0,\infty] \sqcup \mathbb{N} := Z$ *order*



$x \in (0,\infty]$

$n_c \in \mathbb{N}\setminus\{0\}$

$n_c + x = n + x$

addition

$n_c\, x = n\, x$

product

Predomains

Definition(Keimel, 2017):

A predomain is a set S with a transitive binary relation $\prec$ that has the following interpolation property:

Predomains

Definition(Keimel, 2017):

A predomain is a set S with a transitive binary relation $\prec$ that has the following interpolation property:
If F is a finite subset, $F \prec c \implies F \prec b \prec c$ for some $b \in S$.

Predomains

Definition(Keimel, 2017):

A predomain is a set S with a transitive binary relation $\prec$ that has the following interpolation property:
If F is a finite subset, $F \prec c \implies F \prec b \prec c$ for some $b \in S$.

Given a C*-algebra A, we write $[a] \prec [b]$ in $W(A)$ if

Predomains

Definition(Keimel, 2017):

A predomain is a set S with a transitive binary relation $\prec$ that has the following interpolation property:
If F is a finite subset, $F \prec c \implies F \prec b \prec c$ for some $b \in S$.

Given a C*-algebra A, we write $[a] \prec [b]$ in $W(A)$ if there exists $\epsilon > 0$ such that $[a] \leq [(b-\epsilon)_+]$.

Predomains

Definition(Keimel, 2017):

A predomain is a set S with a transitive binary relation $\prec$ that has the following interpolation property:
If F is a finite subset, $F \prec c \implies F \prec b \prec c$ for some $b \in S$.

Given a C*-algebra A, we write $[a] \prec [b]$ in $W(A)$ if there exists $\epsilon > 0$ such that $[a] \leq [(b-\epsilon)_+]$. In this way, $W(A)$ becomes a predomain.