Completeness of the category of Cuntz Semigroups
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TACL 2017
Prague, June 26-30
**The Cuntz Semigroup** is an invariant associated to a C*-algebra $A$, built out of positive elements in $M_\infty(A)$ inspired by Murray von-Neumann equivalence of projections. It has the structure of an ordered commutative monoid.

To use the Cuntz semigroup as a *classification invariant* for C*-algebras, the following aspects would be *desirable*:

- Functoriality.
- Capture the structure of C*-alg. *(add a topology to the ordered semigroup)*
- Preserve the usual categorical constructions for C*-algebras. *(e.g. One typically constructs algebras as $A = \lim_{\to} A_n$ using smaller building blocks $A_n$)*
The original Cuntz semigroup $W(\arrow) \text{ is not a continuous invariant.}$

\[
W(\lim_n M_n(\mathbb{C})) = W(K) = \overline{\mathbb{N}} = \{0, 1, 2, \ldots, \infty\}
\]

while

\[
\lim_n (W(M_n(\mathbb{C}))) = \mathbb{N} = \{0, 1, 2, \ldots\}.
\]
Continuity of the functor $\text{Cu}$

In order to solve this problem Coward, Elliott, and Ivanescu, introduced a stabilized version of the semigroup, and a proper category for the invariant named $\text{Cu}$.

**Theorem (CEI'08)** Given a $C^*$-algebra $A$, $\text{Cu}(A) := W(A \otimes \mathcal{K})$ is an object in $\text{Cu}$, a category with **sequential inductive limits**, and

$$\text{Cu}(\lim_{n} A_n) = \lim_{n} \text{Cu}(A_n).$$
In [APT 14] we introduced a category $W$ of positively ordered monoids with an auxiliary relation, extended the functor $W$ to local $C^*$-algebras, and defined a functor

$$\gamma \colon W \to Cu$$

based on the round ideal completion (Lawson ’97), which is a reflector for the natural embedding $Cu \hookrightarrow \text{PreW}$.
Finally, adding the usual norm completion $\mu$ for algebras, we obtain a commuting diagram with reflector functors allowing to develop arguments in *simpler categories*. As a consequence we obtain, for instance:

**Corollary** The category $\text{Cu}$ has **arbitrary directed limits** and moreover

$$\text{Cu}(\lim_{i \in I} A_i) = \lim_{i \in I} \text{Cu}(A_i).$$
Question
Which limit constructions can be carried out in the category $Cu$? And which of those are preserved by the functor $Cu$?

Theorem (APT)
The category $Cu$ of Cuntz Semigroups is both complete and cocomplete (has both arbitrary small limits and small colimits).

Theorem (?)
Some, yet not all, of these limit constructions are preserved under the functor $Cu$. 
The categories PreW and Q

The Category Cu

A category of positively ordered monoids \((0, +, \leq)\) satisfying

- **O1** Closed under suprema of increasing sequences \(\omega\text{-dcpo.}\)
- **O2** For each \(s \in S\), \(s = \sup s_n, s_n \ll s_{n+1}\ \omega\text{-domain.}\)
- **O3, O4** (\(\ll, \sup, +\) compatibility conditions)

The Category \(\mathcal{P}\)

A category of monoids with a (compatible) transitive relation \(<\).

The category \(W\) (\(\sim\) abstract basis)

- pom with auxiliary relation \(<\)
  - **W2** \((a_n)_n \in a\ll\), \(<\)-cofinal.
  - **W3, W4** (compatibility...)

The category \(Q\)

- pom with auxiliary relation \(<\)
  - **Q1** \(\omega\text{-dcpo.}\)
  - **Q3, Q4** (compatibility...)
Reflection and coreflection

We construct new ordered semigroups using $\prec$-cofinal equivalence classes of certain $\prec$-increasing chains...

...sequences $(s_n)_n$ for semigroups $S$ in $W$, (to add suprema)

$$\gamma(S, \prec) := \{(s_n)_n \mid s_n \in S, s_i \prec s_{i+1}\}$$

...paths for semigroups $P \in Q$, (to add interpolation)

$$\tau(P, \prec) := \{[f] \mid f: (0, 1) \to P, f(\lambda') \prec f(\lambda), \lambda' < \lambda\}$$

Given $S \in W$ and $P \in Q$, $\gamma(S, \prec)$ and $\tau(P, \prec)$ have natural ordered semigroup structures. Moreover...

Using $W_2$ ($\prec$ interpolation). We obtain a universal $W$-map

$$\alpha: S \rightarrow \gamma(S, \prec)$$

$$s \mapsto [(s_n)_n]$$

Using $Q_1$ $\omega$-dcpo, we obtain a universal $Q$-map.

$$\lambda: \tau(P, \prec) \rightarrow P$$

$$[f] \mapsto \sup_n f(1 - \frac{1}{n})$$
Reflection and coreflection

**Theorem**

1. Given $S$ in $W$, $\gamma(S, \prec)$ is a semigroup in $Cu$ and $\gamma: W \to Cu$ is a functor left adjoint to the inclusion functor in $Cu$. (i.e. $Cu$ is a **reflective** subcategory of $W$).

2. Given $P$ in $Q$, $\tau(P, \prec)$ is a semigroup in $Cu$ and $\tau: Q \to Cu$ is a functor right adjoint to the inclusion functor in $Cu$. (i.e. $Cu$ is a **coreflective** subcategory of $Q$).

- $\tau$ can be extended to all $\mathcal{P}$.
- $\tau$ and $\gamma$ coincide in $W \cap Q$.
- $W, Q \subset \mathcal{P}$ not full.
We can now prove that $\text{Cu}$ is both complete and cocomplete, proving the corresponding statements respectively in $Q$ and $W$.

**Example (proof): Existence of coproducts**

- Let $(S_i, i \in I)$ be a family of semigroups in $\text{Cu}$.
- Equip the set 
  \[
  \prod_{i \in I} S_i = \{(s_i) \mid s_i \in S_i \text{ for all } i \in I\}
  \]
  with pointwise order and addition and define an auxiliary relation by pointwise way below:
  \[
  (s_i)_i <_I (t_i)_i \iff s_i \ll t_i \text{ for all } i \in I.
  \]
- It is not difficult to see that $(\prod_{i \in I} S_i, <_I) \in Q$ and is the coproduct in $Q$. Then applying the coreflector
  \[
  \text{Cu} - \prod_{i \in I} S_i = \tau(\prod_{i \in I} S_i, <_I)
  \]
Example (Cu does not preserve inverse limits)

There exists an example due Y. Suzuki of a sequence of $C^*$-algebras $A_n \supseteq A_{n+1}$ such that $A_n \cong O_2$ and such that $\bigcap_n A_n \cong C^*_r(\Gamma)$ for a certain group $\Gamma$.

$Cu(A_n) = \{0, \infty\}$ and $Cu(i_n) = \text{id}$, so $\lim_{\leftarrow} Cu(A_n) = \{0, \infty\}$. But $C^*_r(\Gamma)$ has a trace, so it is Cuntz semigroup can not be $\{0, \infty\}$.

Theorem (Cu preserves inductive limits)

Given an inductive system of $C^*$-algebras and $*$-homomorphisms, $(A_i, f_i)_{i \in I}$, we have

$$Cu(\lim_{i \in I} A_i) = \lim_{i \in I} Cu(A_i).$$

Theorem (Cu preserves coproducts)

Given a family of $C^*$-algebras $(A_i, i \in I)$. Then

$$Cu(\prod_i (A_i)) \cong Cu - \prod_i (Cu(A_i)).$$
Let $\mathcal{U}$ be an ultrafilter on a set $I$, and $(A_i)_{i \in I}$ a family of C*-algebras. The ultraproduct of $(A_i)_{i \in I}$ is defined as

$$\prod_{\mathcal{U}} A_i := \frac{\prod_{i \in I} A_i}{\bigoplus_{\mathcal{U}} A_i}$$

Where $\bigoplus_{\mathcal{U}} A_i$ is the closed ideal of $\prod_{i \in I} A_i$ consisting of tuples of elements whose norm vanish along the ultrafilter.

In categories with limits and colimits one can give a categorical description of ultraproducts:

$$\prod_{\mathcal{U}} A_i \cong \lim_{X \in \mathcal{U}} \left( \prod_{i \in X} A_i \right)$$
Hence, we can do the same for Cuntz semigroups

**Definition**

Given $\mathcal{U}$ an ultrafilter on a set $I$, and $(S_i)_{i \in I}$ a family of Cu-semigroups, we define their ultraproduct as

$$\prod_{\mathcal{U}} S_i := \lim_{X \in \mathcal{U}} \left( \prod_{i \in X} S_i \right)$$

There is an equivalent definition using a certain quotient in the product of semigroups $\prod_{i \in I} S_i$, but using this definition, since Cu preserves arbitrary directed limits and coproducts:

$$\text{Cu}(\prod_{\mathcal{U}} A_i) \cong \text{Cu} - \prod_{\mathcal{U}} \text{Cu}(A_i)$$
Examples

The Cuntz semigroup’s ideal structure coincides with that of the C*-algebra. Let us look at the ideal structure of ultrapowers using Cuntz semigroups:

Consider a non principal ultrafilter \( \mathcal{U} \) on \( \mathbb{N} \):

- If \( S = \{0, \infty\} \), then \( S^\mathcal{U} \cong \{0, \infty\} \).
- If \( S = \{0, 1, \ldots, \infty\} \), then \( S^\mathcal{U} \sim \) Hypernaural numbers (non simple).
- If \( S = [0, \infty] \), then \( S^\mathcal{U} \sim \) Hyperreals (non simple).

In fact, if \( S \) contains a sequence \( s_k \) such that \( 0 \neq 2 \cdot s_{k+1} < s_k \), then \( S^\mathcal{U} \) contains infinitesimals (non simple).

(L. Robert) proved that if \( A \) is simple, non purely infinite and non elementary C*-algebra (equiv. \( \text{Cu}(A) \neq \mathbb{N} \)), a sequence as the one above exists.

Hence

\[ A^\mathcal{U} \] is simple, if and only if \( A \) is purely infinite.