Completeness of the category of Cuntz Semigroups

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TACL 2017
Prague, June 26-30

Introduction, motivation and aim.

The Cuntz Semigroup is an invariant associated to a C*-algebra A, built out of positive elements in $M_\infty(A)$ inspired by Murray von-Neumann equivalence of projections. It has the structure of an ordered commutative monoid.

To use the Cuntz semigroup as a *classifcation invariant* for C*-algebras, the following aspects would be *desirable*:

- Functoriality.
- Capture the structure of C*-alg. (*add a topology to the ordered semigroup*)
- Preserve the usual categorical constructions for C*-algebras. (*e.g. One typicall constructs algebras as $A = \lim_{\rightarrow} A_n$ using smaller building blocks A_n*)

Continuity of the functor Cu

$$\text{C}^*\text{-algebras} \xrightarrow{\mathrm{W}} \text{Semigroups}$$

The original Cuntz semigroup W(−) is not a continuous inavariant.

$$\mathrm{W}(\lim_n M_n(\mathbb{C})) = \mathrm{W}(\mathcal{K}) = \overline{\mathbb{N}} = \{0, 1, 2, \ldots, \infty\}$$

while

$$\lim_n(\mathrm{W}(M_n(\mathbb{C}))) = \mathbb{N} = \{0, 1, 2, \ldots\}.$$

Continuity of the functor Cu

$$\begin{tikzcd} \mathrm{C}^*\text{-algebras} \arrow[r, "\mathrm{W}"] \arrow[dr, "\mathrm{Cu}"] & \text{Semigroups} \\ & \mathrm{Cu}\text{-semigroups} \end{tikzcd}$$

In order to solve this problem Coward, Elliott, and Ivanescu, introduced a stabilized version of the semigroup, and a proper category for the invariant named Cu.

Theorem (CEI'08) Given a C^*-algebra A, $\mathrm{Cu}(A) := \mathrm{W}(A \otimes \mathcal{K})$ is an object in Cu, a category with sequential inductive limits, and

$$\mathrm{Cu}(\lim_n A_n) = \lim_n \mathrm{Cu}(A_n).$$

Continuity of the functor Cu

$$\begin{tikzcd}
\overset{\text{local } C^*\text{-algebras}}{\cancel{C^*\text{-algebras}}} \arrow[r, "\mathrm{W}"] \arrow[dr, "\mathrm{Cu}"] & \overset{\mathrm{W}\text{-semigroups}}{\cancel{\text{Semigroups}}} \arrow[d, dotted, "\gamma?"] \\
 & \mathrm{Cu}\text{-semigroups}
\end{tikzcd}$$

In [APT 14] we introduced a category W of positively ordered monoids with an auxiliary relation, extended the functor W to local C*-algebras, and defined a functor

$$\gamma\colon \mathrm{W} \to \mathrm{Cu}$$

based on the round ideal completion (Lawson '97), which is a reflector for the natural embedding $\mathrm{Cu} \hookrightarrow \mathrm{PreW}$.

Continuity of the functor Cu

$$
\begin{tikzcd}
\text{local } C^*\text{-algebras} \arrow[r, "W"] \arrow[d, "\mu"] & W\text{-semigroups} \arrow[d, "\gamma"'] \\
C^*\text{-algebras} \arrow[r, "\mathrm{Cu}"] \arrow[u, bend left, dotted] & \mathrm{Cu}\text{-semigroups} \arrow[u, bend right, dotted]
\end{tikzcd}
$$

Finally, adding the usual norm completion μ for algebras, we obtain a commuting diagram with reflector functors allowing to develop arguments in *simpler categories*. As a consequence we obtain, for instance:

Corollary The category Cu has arbitrary directed limits and moreover

$$\mathrm{Cu}(\lim_{i\in I} A_i) = \lim_{i\in I} \mathrm{Cu}(A_i).$$

Question

Which limit constructions can be carried out in the category Cu? And which of those are preserved by the functor Cu?

Theorem (APT)

The category Cu of Cuntz Semigroups is both complete and cocomplete (has both arbitrary small limits and small colimits)

Theorem (?)

Some, yet not all, of these limit constructions are preserved under the functor Cu.

The categories PreW and Q

The Category Cu

A category of *positively ordered monoids* $(0, +, \leq)$ satisfying

- O1 Closed under suprema of increasing sequences ω-dcpo.
- O2 For each $s \in S$, $s = \sup s_n$, $s_n \ll s_{n+1}$ ω-domain.
- O3,O4 $(\ll, \sup, +$ compatibility conditions)

The Category $\mathcal{P}$

A category of monoids with a (compatible) transitive relation $\prec$.

The category W ($\sim$ abstract basis)

pom with auxiliary relation $\prec$

- W2 $(a_n)_n \in a^{\prec}$, $\prec$-cofinal.
- W3,W4 (compatibility...)

The category $\mathcal{Q}$

pom with auxiliary relation $\prec$

- Q1 ω–dcpo.
- Q3,Q4 (compatibility...)

Reflection and coreflection

We construct new orderd semigroups using $\prec$-cofinal equivalence classes of certain $\prec$-increasing chains...

...sequences $(s_n)_n$ for semigroups S in W, (to add suprema)

$$\gamma(S,\prec) := \{[(s_n)_n] \text{ where } s_n \in S, s_i \prec s_{i+1}\}$$

...paths for semigroups $P \in Q$, (to add interpolation)

$$\tau(P,\prec) := \{[f] \text{ where } f\colon (0,1) \to P, f(\lambda') \prec f(\lambda), \lambda' < \lambda\}$$

Given $S \in$ W and $P \in Q$, $\gamma(S,\prec)$ and $\tau(P,\prec)$ have natural ordered semigroup structures. Moreover...

Using W2 ($\prec$ interpolation). We obatin a *universal* W-map

$$\begin{array}{cccc} \alpha\colon & S & \longrightarrow & \gamma(S,\prec) \\ & s & \longmapsto & [(s_n)_n] \end{array}$$

Using Q1 ω-dcpo, we obtain a *universal* Q-map.

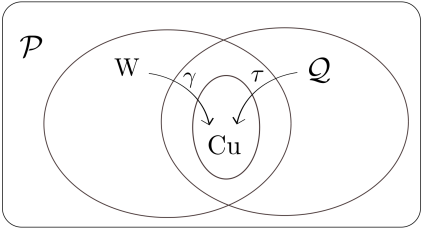
$$\begin{array}{cccc} \lambda\colon & \tau(P,\prec) & \longrightarrow & P \\ & [f] & \longmapsto & \sup_n f(1-\frac{1}{n}) \end{array}$$

Reflection and coreflection

Theorem

1 Given S in W, $\gamma(S, \prec)$ is a semigroup in Cu and $\gamma\colon W \to \mathrm{Cu}$ is a functor left adjoint to the inclusion functor in Cu. (i.e. Cu is a reflective subcategory of W).

2 Given P in $\mathcal{Q}$, $\tau(P, \prec)$ is a semigroup in Cu and $\tau\colon \mathcal{Q} \to \mathrm{Cu}$ is a functor right adjoint to the inclusion functor in Cu. (i.e. Cu is a coreflective subcategory of $\mathcal{Q}$).



- τ can be extended to all $\mathcal{P}$.
- τ and γ coincide in $W \cap \mathcal{Q}$.
- $W, \mathcal{Q} \subset \mathcal{P}$ not full.

Completeness and Cocompleteness

We can now prove that Cu is both *complete and cocomplete*, proving the corresponding statements respectively in $\mathcal{Q}$ and W.

Example (proof): *Existence of coproducts*

- Let $(S_i, i \in I)$ be a family of semigroups in Cu.

- Equip the *set*

$$\prod_{i \in I} S_i = \{(s_i) \mid s_i \in S_i \text{ for all } i \in I\}$$

with *pointwise* order and addition and define an auxiliary relation by *pointwise* way below:

$$(s_i)_i \prec_I (t_i)_i \Leftrightarrow s_i \ll t_i \text{ for all } i \in I.$$

- It is not difficult to see that $(\prod_{i \in I} S_i, \prec_I) \in \mathcal{Q}$ and is the coproduct in $\mathcal{Q}$. Then applying the coreflector

$$\mathrm{Cu} - \prod_{i \in I} S_i = \tau(\prod_{i \in I} S_i, \prec_I)$$

Example (Cu does not preserve inverse limits)

There exists an example due Y. Suzuki of a sequence of C*-algebras $A_n \supseteq A_{n+1}$ such that $A_n \cong \mathcal{O}_2$ and such that $\bigcap_n A_n \cong C_r^*(\Gamma)$ for a certain group Γ.

$\mathrm{Cu}(A_n) = \{0, \infty\}$ and $\mathrm{Cu}(i_n) = \mathrm{id}$, so $\lim_{\leftarrow} \mathrm{Cu}(A_n) = \{0, \infty\}$. But $C_r^*(\Gamma)$ has a trace, so it is Cuntz semigroup can not be $\{0, \infty\}$.

Theorem (Cu preserves inductive limits)

Given an inductive system of C*-algebras and $*$-homomorphisms, $(A_i, f_i)_{i\in I}$, we have

$$\mathrm{Cu}(\lim_{i\in I} A_i) = \lim_{i\in I} \mathrm{Cu}(A_i).$$

Theorem (Cu preserves coproducts)

Given a family of C*-algebras $(A_i, i \in I)$. Then

$$\mathrm{Cu}(\prod_i (A_i)) \cong \mathrm{Cu} - \prod_i (\mathrm{Cu}(A_i))$$

Application: Ultraproducts

Let $\mathcal{U}$ be an ultrafilter on a set I, and $(A_i)_{i\in I}$ a family of C*-algebras. The *ultraproduct* of $(A_i)_{i\in I}$ is defined as

$$\prod_{\mathcal{U}} A_i := \frac{\prod_{i\in I} A_i}{\bigoplus_{\mathcal{U}} A_i}$$

Where $\bigoplus_{\mathcal{U}} A_i$ is the closed ideal of $\prod_{i\in I} A_i$ consisting of tuples of elements whose norm vanish along the ultrafilter.
In categories with limits and colimits one can give a categorical description of *ultraproducts*:

$$\prod_{\mathcal{U}} A_i \cong \lim_{X\in\mathcal{U}} \Big(\prod_{i\in X} A_i\Big)$$

Hence, we can do the same for Cuntz semigroups

Definition

Given $\mathcal{U}$ an ultrafilter on a set I, and $(S_i)_{i\in I}$ a family of Cu-semigroups, we define their *ultraproduct* as

$$\prod_{\mathcal{U}} S_i := \lim_{X\in\mathcal{U}} (\prod_{i\in X} S_i)$$

There is an equivalent definition using a certain quotient in the product of semigroups $\prod_{i\in I} S_i$, but using this definition, since Cu preserves arbitrary directed limits and coproducts:

$$\mathrm{Cu}(\prod_{\mathcal{U}} A_i) \cong \mathrm{Cu}-\prod_{\mathcal{U}} \mathrm{Cu}(A_i)$$

Examples

The Cuntz semigroup's ideal structure coincides with that of the C*-algebra. Let us look at the ideal structure of ultrapowers using Cuntz semigroups:
Consider a non principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$

If $S = \{0, \infty\}$, then $S^{\mathcal{U}} \cong \{0, \infty\}$

If $S = \{0, 1, \ldots, \infty\}$, then $S^{\mathcal{U}} \sim$ Hypernaural numbers (non simple).

If $S = [0, \infty]$, then $S^{\mathcal{U}} \sim$ Hyperreals (non simple).

In fact, if S contains a sequence s_k such that $0 \neq 2 * s_{k+1} < s_k$, then $S^{\mathcal{U}}$ contains infinitessimal elements (non simple).

(L. Robert) proved that if A is simple, non purely infinite and non elementary C*-algebra (equiv. $\mathrm{Cu}(A) \neq \mathbb{N}$), a sequence as the one above exists.
Hence

$$A^{\mathcal{U}} \text{ is simple, if and only if } A \text{ is purely infinite}$$