Topology as faithful communication through relations

Samuele Maschio and Giovanni Sambin

UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Matematica
Università di Padova

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$$(\mathcal{M}_X, \sim_X) \underset{\Delta}{\overset{\nabla}{\rightleftarrows}} (\mathcal{M}_S, \sim_S)$$

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if $\nabla(\Delta(m)) \sim_X m$,

then the communication of m is **faithful**

i. e. m is **faithfully communicable**.

Communication and topology

Goal:
to give a characterization of basic topological notions
in terms of **faithfully communicable** notions.

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1. X represents **points**
2. S a set of indexes for a **basis of neighbourhoods** of a topology on X
3. $\Vdash$ relation from X to S, $x \Vdash a$: "x is in the neighbourhood indexed by a".

   a is the index of the neighbourhood $\mathrm{ext}\, a := \{x \in X | x \Vdash a\}$

   $\Diamond x := \{a \in S | x \Vdash a\}$.

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rest, $\Box$ approximation by defect

ext (resp. $\Diamond$) is left adjoint to $\Box$ (resp. rest).

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$D \equiv$ communicable message in the following system with X, S individuals

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Other decoding procedures (1)

What about these decoding procedures?

$$(\mathcal{P}(X), =_X) \underset{\Box}{\overset{\mathsf{rest}}{\rightleftarrows}} (\mathcal{P}(S), =_S) \qquad (\mathcal{P}(X), =_X) \underset{\Diamond}{\overset{\mathsf{ext}}{\rightleftarrows}} (\mathcal{P}(S), =_S)$$

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The inverse statement does not hold:
in $(\mathbf{2}, \Vdash, \mathbf{3})$ where $x \Vdash a \equiv^{def} x = a \vee a = 2$,
singletons $\{x\}$ are clopen, but $\text{ext}\Diamond\{x\} = \mathbf{2}$ and $\text{rest}\Box\{x\} = \varnothing$.

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$D^{\rightarrow} := \{a \in S \mid D \subseteq \text{ext}\, a\}$ and $U^{\leftarrow} := \{x \in X \mid U \subseteq \Diamond x\}$ (**polarities** in Birkhoff)

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	ext	rest	$\leftarrow$
$\Diamond$	$\varnothing$, X	Closed sets	T_0-points inters. of open sets
$\square$	Open sets	$\varnothing$, X	$\varnothing$
$\rightarrow$	X	closed T_0-points	intersections of open sets

Communication of relations

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Individuals are (X, Y) and (S, T)
Messages are $\mathrm{Rel}(X, Y)$ and $\mathrm{Rel}(S, T)$.

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i. e. if for every b, $r^{-}\mathrm{ext}\, b = \mathrm{ext}\Box r^{-}\mathrm{ext} b$ is open.

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$$(\mathrm{Rel}(X, Y), \sim_{(X,Y)}) \underset{\sigma}{\overset{\rho}{\leftrightarrows}} (\mathrm{Rel}(S, T), \sim_{(S,T)})$$

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$$\begin{tikzcd}(\mathrm{Rel}(X,Y), \sim_{(X,Y)}) \arrow[r, shift right, "\sigma"'] & (\mathrm{Rel}(S,T), \sim_{(S,T)}) \arrow[l, shift right, "\rho"']\end{tikzcd}$$

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Then r is communicable if and only if r is continuous.

Continuous relations as commutative diagram

Notice that $r : X \to Y$ is continuous from $\mathcal{X}$ to $\mathcal{Y}$ if and only if there exists $s : S \to T$ such that the following diagram commutes in Rel:

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Equivalent continuos relations corresponds to squares with equal diagonals. Basic pairs and continuous relations form a category which is equivalent to the Freyd completion of Rel.

Concrete spaces

A **concrete space** is a basic pair $(X, \Vdash, S)$ such that

1. $X = \mathrm{ext} S$
2. for all $a, b \in S$, $\mathrm{ext}\, a \cap \mathrm{ext}\, b = \bigcup\{\mathrm{ext}\, c \mid \mathrm{ext}\, c \subseteq \mathrm{ext}\, a \cap \mathrm{ext}\, b\}$

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Thank you for your attention!