Locally tabular polymodal logics

Ilya Shapirovsky

Institute for Information Transmission Problems of the Russian Academy of Sciences, Moscow

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Locally tabular (or *locally finite*) logics

A logic L is *locally tabular* if, for any finite k, there exist only finitely many pairwise nonequivalent formulas in L built from the variables $p_1, \ldots, p_k$.

Equivalently, a logic L is locally tabular if the variety of its algebras is *locally finite*, i.e., every finitely generated L-algebra is finite.

- If a logic is locally tabular, then it has the finite model property (thus, it is Kripke complete).
- Every extension of a locally tabular logic is locally tabular (thus, it has the finite model property).
- Every finitely axiomatizable extension of a locally tabular logic is decidable.

Local tabularity above $\mathrm{K4}$

If $\mathrm{L} \supseteq \mathrm{K4}$,

L is locally tabular iff it is a logic of *finite height*.

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- Every locally tabular logic is a logic of *finite height*.
- Every locally tabular logic is *pretransitive* (that is, the master modality is expressible).
- There is a natural characterization of local tabularity in terms of partitions of clusters, occurring in frames of the logic.
- For extensions of logics much weaker than $\mathrm{K4}$, finite height is sufficient for (thus, equivalent to) local tabulararity.

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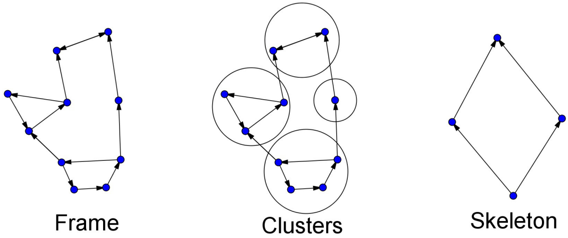
The aim of this talk is to extend these results for the polymodal case, and then to discuss some corollaries and open problems.

Unimodal case. Frames of finite height

A poset F is of *finite height* $\leq h$ if its every chain contains at most h elements.

R^* denotes the transitive reflexive closure of R.
$\sim_R = R^* \cap R^{*-1}$, an equivalence class modulo $\sim_R$ is a *cluster* in (W, R) (so clusters are maximal subsets where R^* is universal).
The *skeleton of* (W, R) is the poset $(W/{\sim_R}, \leq_R)$, where for clusters $C, \ D$, $C \leq_R D$ iff xR^*y for some $x \in C, y \in D$.

Height of a frame is the height of its skeleton.



Formulas of finite height (transitive case):

$$B_1 = p_1 \to \Box\Diamond p_1, \quad B_{i+1} = p_{i+1} \to \Box(\Diamond p_{i+1} \vee B_i)$$

Proposition

B_h is valid in a transitive F iff the height of $\mathrm{F} \leq h$.

A logic $\mathrm{L} \supseteq \mathrm{K4}$ is of *finite height* if it contains B_h for some h.

Theorem (Segerberg, 1971; Maksimova, 1975)

For a logic $\mathrm{L} \supseteq \mathrm{K4}$,

$$\mathrm{L} \text{ is locally tabular iff it is of finite height.}$$

A logic L pretransitive if the master modality is expressible in L. Formally:

A logic L is said to be *pretransitive* (or *conically expressive*), if there exists a formula $\chi(p)$ with a single variable p such that for every Kripke model M with $\mathrm{M} \vDash \mathrm{L}$ and for every w in M we have:

$$\mathrm{M}, w \vDash \chi(p) \iff \forall u(wR^* u \Rightarrow \mathrm{M}, u \vDash p).$$

L is *m-transitive* iff $\mathrm{L} \vdash \Diamond^{m+1} p \to \Diamond^{\leq m} p$ for some $m \geq 0$, where $\Diamond^{0}\varphi := \varphi,\ \Diamond^{i+1}\varphi := \Diamond\Diamond^{i}\varphi,\ \Diamond^{\leq m}\varphi := \bigvee_{i=0}^{m} \Diamond^{i}\varphi,\ \Box^{\leq m}\varphi := \neg\Diamond^{\leq m}\neg\varphi.$

Theorem (Shehtman, 2009)

- L *is pretransitive iff it is m-transitive for some $m \geq 0$.*
- *For an m-transitive logic L, the set $\{\varphi \mid \mathrm{L} \vdash \varphi^{[m]}\}$ is a logic containing $\mathrm{S4}$.*

Here $\varphi^{[m]}$ denotes the formula obtained from φ by replacing $\Diamond$ with $\Diamond^{\leq m}$ and $\Box$ with $\Box^{\leq m}$.

Theorem (Shehtman, Sh, 2016)

Every locally tabular logic is a pretransitive logic of finite height:
L *is locally tabular* $\Rightarrow \mathrm{L}$ *contains* $(\Diamond^{m+1}p \to \Diamond^{\leq m}p) \wedge B_h^{[m]}$ *for some* m, h.

Formulas of finite height (transitive case):
$B_1 = p_1 \to \Box\Diamond p_1, \quad B_{i+1} = p_{i+1} \to \Box(\Diamond p_{i+1} \vee B_i)$

Formulas of finite height (pretransitive case):
$B_h^{[m]}$ is obtained from B_h by replacing $\Diamond$ with $\Diamond^{\leq m}$ and $\Box$ with $\Box^{\leq m}$.

$R^{\leq m} = \bigcup\limits_{0 \leq i \leq m} R^i$, where $R^0 = Id(W)$, $R^{i+1} = R \circ R^i$.
R is *m-transitive*, if $R^{\leq m} = R^*$, or equivalently, $R^{m+1} \subseteq R^{\leq m}$.

Proposition. R is m-transitive iff $(W, R) \vDash \Diamond^{m+1}p \to \Diamond^{\leq m}p$.
Proposition. For an m-transitive frame F, $\mathrm{F} \vDash B_h^{[m]} \iff ht(\mathrm{F}) \leq h$.

Unimodal case. Necessary condition.

L is locally tabular $\Rightarrow$ L contains $(\Diamond^{m+1}p \to \Diamond^{\leq m}p) \wedge B_h^{[m]}$ for some m, h.

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Theorem (Kudinov, Sh, 2015) All the logics $\mathrm{K} + (\Diamond^{m+1}p \to \Diamond^{\leq m}p) \wedge B_h^{[m]}$ have the FMP.

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Theorem (Miyazaki, 2004) (Kostrzycka, 2008) For $m \geq 2$, all the above logics have Kripke incomplete extensions.

Corollary For $m \geq 2$, none of them are locally tabular.

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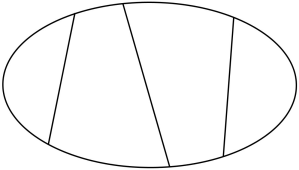
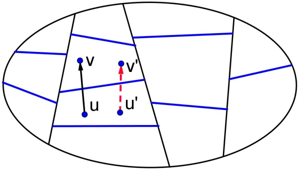
Problem, 1960s

For $m > 1$, pretransitive logics are very complex and not well-studied. In particular, the FMP (and even the decidability) of the logics $\mathrm{K} + \Diamond^{m+1} p \to \Diamond^{\leq m} p$ is unknown for $m > 1$.
(*"Perhaps one of the most intriguing open problems in Modal Logic"* [Wolter F., Zakharyaschev M. Modal decision problems // Handbook of Modal Logic. 2007].)

A partition $\mathcal{A}$ of $\mathrm{F} = (W, R)$ is R-*tuned*, if for any $U, V \in \mathcal{A}$

$$\exists u \in U \; \exists v \in V \; uRv \quad \Rightarrow \quad \forall u \in U \; \exists v \in V \; uRv.$$





Proposition $\mathcal{A}$ is R-tuned iff $\{\cup x \mid x \subseteq \mathcal{A}\}$ forms a subalgebra of the modal algebra of (W, R).

Unimodal case. Semantic criterions.

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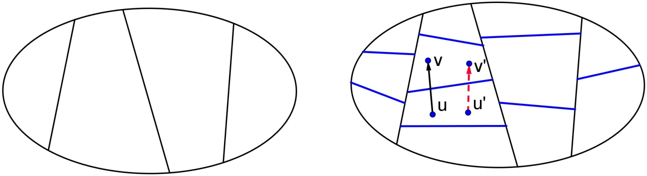

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Proposition (Franzen, Fine, 1970s)

If for every finite partition $\mathcal{A}$ of W there exists a finite R-tuned refinement $\mathcal{B}$ of $\mathcal{A}$, then $Log(W, R)$ has the FMP.

$Log(\mathbb{N}, \leq)$ **has the FMP:** Refine $\mathcal{A}$ in such a way that all elements of $\mathcal{B}$ are infinite or singletons, and singletons cover an initial segment of $\mathbb{N}$.

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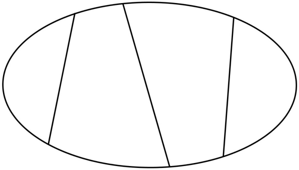

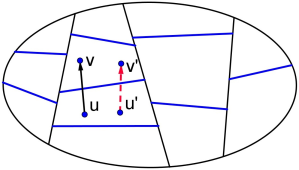

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$Log(\mathbb{N}, \leq)$ **has the FMP:** Refine $\mathcal{A}$ in such a way that all elements of $\mathcal{B}$ are infinite or singletons, and singletons cover an initial segment of $\mathbb{N}$.
$Log(\mathbb{N}, \leq)$ **is not locally tabular:** $(\mathbb{N}, \leq)$ is of infinite height.

A partition $\mathcal{A}$ of $\mathrm{F} = (W, R)$ is *R-tuned*, if for any $U, V \in \mathcal{A}$

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A class of frames $\mathcal{F}$ is *ripe*, if there exists $f : \mathbb{N} \to \mathbb{N}$ s.t. for every finite partition $\mathcal{A}$ of a frame $(W, R) \in \mathcal{F}$ there exists an R-tuned refinement $\mathcal{B}$ of $\mathcal{A}$ such that $|\mathcal{B}| \leq f(|\mathcal{A}|)$.

For a class $\mathcal{F}$ of frames let $cl\mathcal{F}$ be the class of clusters occurring in frames from $\mathcal{F}$: $cl\mathcal{F} = \{\mathrm{F}{\restriction}C \mid \mathrm{F} \in \mathcal{F} \text{ and } C \text{ is a cluster in } \mathrm{F}\}$.

A class $\mathcal{F}$ of frames is of *finite height* if $\exists h \in \mathbb{N}$ s.t. $ht(\mathrm{F}) \leq h$ for all $\mathrm{F} \in \mathcal{F}$.

Theorem (Shehtman, Sh, 2016)

$Log\, \mathcal{F}$ is locally tabular iff $\mathcal{F}$ is ripe iff $\mathcal{F}$ is of finite height and $cl\mathcal{F}$ is ripe.

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Theorem (Segerberg, Maksimova, 1970s). For a logic $\mathrm{L} \supseteq \mathrm{K4}$,

L is locally tabular iff it is of finite height.

Proof. If (W, R) is a cluster in a transitive frame, than any partition of (W, R) is R-tuned.

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Theorem. For a logic $\mathrm{L} \supseteq \mathrm{wK4} = \mathrm{K} + \Diamond\Diamond p \to \Diamond p \vee p$,
L is locally tabular iff it is of finite height.

Proof If (W, R) is a cluster in a $\mathrm{wK4}$-frame, than any partition of (W, R) is R-tuned.

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Theorem (Shehtman, Sh, 2016)

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Theorem. For $m > 0$ and a logic L containing $\Diamond^{m+1}p \to \Diamond p \vee p$,
L is locally tabular iff it is of finite height.

Proof The class of clusters in frames validating $\Diamond^{m+1}p \to \Diamond p \vee p$ is ripe.

Polymodal case

Fix some $n > 0$ and the n-modal language with the modalities $\Diamond_0, \ldots, \Diamond_{n-1}$.

For a Kripke frame $\mathrm{F} = (W, (R_i)_{i<n})$, put

$$R_{\mathrm{F}} = \cup_{i<n} R_i.$$

$\sim_{\mathrm{F}}$ is the equivalence relation $R_{\mathrm{F}}^* \cap R_{\mathrm{F}}^{*\,-1}$, where R_{F}^* is the transitive reflexive closure of R_{F}.
A *cluster* in F is an equivalence class under $\sim_{\mathrm{F}}$.
The *height* of F is the height of (W, R_{F}).
F is *m-transitive* if R_{F} is m-transitive.

Theorem

Every locally tabular n-modal logic is a pretransitive logic of finite height:
L *is locally tabular* $\Rightarrow \mathrm{L}$ *contains* $(\Diamond^{m+1}p \to \Diamond^{\leq m}p) \wedge B_h^{[m]}$ *for some* m, h.

Master modality (polymodal case)

$\Diamond^{\leq m}\varphi = \bigvee_{i \leq m} \Diamond^i \varphi$, $\Box^{\leq m}\varphi = \neg\Diamond^{\leq m}\neg\varphi$, where

$$\Diamond\varphi = \Diamond_0\varphi \vee \ldots \vee \Diamond_{n-1}\varphi$$

Formulas of finite height (polymodal case):

$B_1^{[m]} = p_1 \to \Box^{\leq m}\Diamond^{\leq m}p_1$, $B_{i+1}^{[m]} = p_{i+1} \to \Box^{\leq m}(\Diamond^{\leq m}p_{i+1} \vee B_i^{[m]})$.

Proposition. F is m-transitive iff $\mathrm{F} \models \Diamond^{m+1}p \to \Diamond^{\leq m}p$.
Proposition. For an m-transitive frame F, $\mathrm{F} \models B_h^{[m]} \iff ht(\mathrm{F}) \leq h$.

Let $\mathrm{F} = (W, (R_i)_{i<n})$ be a Kripke frame. A partition $\mathcal{A}$ of W is F-*tuned*, if for every $U, V \in \mathcal{A}$, and every $i < n$

$$\exists u \in U \; \exists v \in V \;\; uR_i v \quad \Rightarrow \quad \forall u \in U \; \exists v \in V \;\; uR_i v.$$

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A class of frames $\mathcal{F}$ is *ripe*, if there exists $f : \mathbb{N} \to \mathbb{N}$ s.t. for every finite partition $\mathcal{A}$ of a frame $\mathrm{F} \in \mathcal{F}$ there exists an F-tuned refinement $\mathcal{B}$ of $\mathcal{A}$ such that $|\mathcal{B}| \leq f(|\mathcal{A}|)$.

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Theorem (Main result)

$Log\,\mathcal{F}$ is locally tabular iff $\mathcal{F}$ is ripe iff $\mathcal{F}$ is of finite height and $cl\,\mathcal{F}$ is ripe.

Example

K_n is the least n-modal logic, $\Box\varphi = \Box_0\varphi \wedge \ldots \wedge \Box_{n-1}\varphi$.

Theorem (Gabbay, Shehtman, 1998; Shehtman, 2014).
For all $h \geq 0$, $\mathrm{K}_n + \Box^h\bot$ is locally tabular.

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Proof.

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Proof. The logic is Kripke complete, the height of its frames $\leq h+1$,

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K_n is the least n-modal logic, $\Box\varphi = \Box_0\varphi \wedge \ldots \wedge \Box_{n-1}\varphi$.

Theorem (Gabbay, Shehtman, 1998; Shehtman, 2014).
For all $h \geq 0$, $\mathrm{K}_n + \Box^h\bot$ is locally tabular.

Proof. The logic is Kripke complete, the height of its frames $\leq h+1$, their clusters are singletons.

Polymodal case. Semantic criterions.

Let $\mathrm{F} = (W, (R_i)_{i<n})$ be a Kripke frame. A partition $\mathcal{A}$ of W is F-*tuned*, if for every $U, V \in \mathcal{A}$, and every $i < n$

$$\exists u \in U\ \exists v \in V\ uR_i v \quad \Rightarrow \quad \forall u \in U\ \exists v \in V\ uR_i v.$$

A class of frames $\mathcal{F}$ is *ripe*, if there exists $f : \mathbb{N} \to \mathbb{N}$ s.t. for every finite partition $\mathcal{A}$ of a frame $\mathrm{F} \in \mathcal{F}$ there exists an F-tuned refinement $\mathcal{B}$ of $\mathcal{A}$ such that $|\mathcal{B}| \leq f(|\mathcal{A}|)$.

Theorem (Main result)

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For $\mathrm{F} = (W, (R_i)_{i<n})$, let $\mathrm{F}_u = (W, (R_i)_{i<n}, W \times W)$.
For a class $\mathcal{F}$ of frames put $\mathcal{F}_u = \{\mathrm{F}_u \mid \mathrm{F} \in \mathcal{F}\}$.

For an n-modal logic L, let L_u be the $(n+1)$-modal logic in the language with the extra modality $[\mathrm{u}]$, that is the extension of L with the $\mathrm{S5}$-axioms for $[\mathrm{u}]$ and the axioms $[\mathrm{u}]p \to \Box_i p$ for all $i < n$.

Proposition. $\mathrm{F} \vDash \mathrm{L}$ iff $\mathrm{F}_u \vDash \mathrm{L}_u$.
Proposition. If $\mathcal{F}$ is the class of all the frames of L and L_u is Kripke complete, then $\mathrm{L}_u = \mathit{Log}\,\mathcal{F}_u$.

Extending a logic with $[\mathrm{u}]$ is not safe: we can lose the FMP (Wolter, 1994), decidability (Spaan, 1993), Kripke completeness (Kracht, 1999).

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Theorem

If L is locally tabular, then L_u is locally tabular.

Proof.

Trivially, if a partition is tuned for a frame $(W, (R_i)_{i<n})$, then it is tuned for the frame $(W, (R_i)_{i<n}, W \times W)$. $\square$

For $\mathrm{F} = (W, (R_i)_{i<n})$, put $\mathrm{F}_t = (W, (R_i)_{i<n}, (R_i^{-1})_{i<n})$.

For an n-modal logic L, let L_t be a $2n$-modal logic, which is the extension of L with the axioms $p \to \Box_i \Diamond_i^{-1} p$ and $p \to \Box_i^{-1} \Diamond_i p$ for all $i < n$.

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Tuned partitions allows us to construct *(minimal) filtrations*. Thus, locally tabular logics *admit filtration*.

Theorem (Kikot, Zolin, Sh, 2014). If L admits filtration and L_t is Kripke complete, then L_t has the FMP.

Theorem

If L is locally tabular, then L_t has the finite model property.

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Problem

Which properties of L_1 and L_2 guarantee the local tabularity of $\mathrm{L}_1 \times \mathrm{L}_2$?
What are the properties of $\mathrm{L}_1 \times \mathrm{L}_2$, if L_1 and L_2 are locally tabular?

Log $\mathcal{F}$ is locally tabular iff $\mathcal{F}$ is ripe iff $\mathcal{F}$ is of finite height and *cl*$\mathcal{F}$ is ripe.

Corollary. Suppose L_0 is a (weakly) canonical pretransitive logic and the class of clusters occurring in its frames is ripe. Then for any logic $\mathrm{L} \supseteq \mathrm{L}_0$,

L is locally tabular iff L is of finite height.

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Corollary. L is locally tabular iff L is the logic of some class $\mathcal{F}$ such that $\mathcal{F}$ is of finite height and *Log cl*$\mathcal{F}$ is locally tabular.

In the above, *Log cl*$\mathcal{F}$ is a pretransitive logic of height 1 (the master modality satisfies $\mathrm{S5}$).

Theorem (Kowalski, 2006). L is a pretransitive logic of height 1 iff *Alg* L is a discriminator variety.

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Question. What is the finite height algebraically?

Theorem (Maksimova, 1975).

A logic $\mathrm{L} \supseteq \mathrm{K4}$ is locally tabular iff its 1-generated free algebra $\mathfrak{A}_{\mathrm{L}}(1)$ is finite.

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Proof.

Recall that in this case L is locally tabular iff L is of finite height.

The $\Box^*$-fragment of L is a logic L_* containing $\mathrm{S4}$.
If $\mathfrak{A}_{\mathrm{L}}(1)$ is finite, then $\mathfrak{A}_{\mathrm{L}_*}(1)$ is finite, thus, L_* is locally tabular, thus, it is of finite height. Thus, L is of finite height. $\Box$

Problem (1970s).

Does this equivalence hold for every modal logic?

Concluding remarks

Theorem (Maksimova, 1975).

A logic $\mathrm{L} \supseteq \mathrm{K4}$ is locally tabular iff its 1-generated free algebra $\mathfrak{A}_{\mathrm{L}}(1)$ is finite.

Corollary

Suppose L_0 is a canonical pretransitive logic and the class of clusters occurring in its frames is ripe. Then for any logic $\mathrm{L} \supseteq \mathrm{L}_0$,

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Question. Suppose that we know how to tune all two-element partitions of a cluster. Can we tune all finite ones?

Thank you!