Locally tabular polymodal logics

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A logic $L$ is *locally tabular* if, for any finite $k$, there exist only finitely many pairwise nonequivalent formulas in $L$ built from the variables $p_1, \ldots, p_k$.

Equivalently, a logic $L$ is locally tabular if the variety of its algebras is *locally finite*, i.e., every finitely generated $L$-algebra is finite.

- If a logic is locally tabular, then it has the finite model property (thus, it is Kripke complete).
- Every extension of a locally tabular logic is locally tabular (thus, it has the finite model property).
- Every finitely axiomatizable extension of a locally tabular logic is decidable.
Local tabularity above $K_4$

If $L \supseteq K_4$,

$L$ is locally tabular iff it is a logic of *finite height*.

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Every locally tabular logic is a logic of *finite height*.

Every locally tabular logic is *pretransitive* (that is, the master modality is expressible).

There is a natural characterization of local tabularity in terms of *partitions of clusters*, occurring in frames of the logic.

For extensions of logics much weaker than \( K_4 \), finite height is sufficient for (thus, equivalent to) local tabulararity.

(Shehtman, Sh, 2016)
Local tabularity above $\text{K4}$

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Local tabularity above $\text{K}$

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- Every locally tabular logic is *pretransitive* (that is, the master modality is expressible).
- There is a natural characterization of local tabularity in terms of *partitions of clusters*, occurring in frames of the logic.
- For extensions of logics much weaker than $\text{K4}$, finite height is sufficient for (thus, equivalent to) local tabulararity.

(Shehtman, Sh, 2016)

The aim of this talk is to extend these results for the polymodal case, and then to discuss some corollaries and open problems.
A poset $F$ is of \textit{finite height} $\leq h$ if its every chain contains at most $h$ elements.

$R^*$ denotes the transitive reflexive closure of $R$. $\sim_R = R^* \cap R^*^{-1}$, an equivalence class modulo $\sim_R$ is a \textit{cluster} in $(W, R)$ (so clusters are maximal subsets where $R^*$ is universal).

The \textit{skeleton of} $(W, R)$ is the poset $(W/\sim_R, \leq_R)$, where for clusters $C, D$, $C \leq_R D$ iff $xR^*y$ for some $x \in C, y \in D$.

\textit{Height of a frame} is the height of its skeleton.
Formulas of finite height (transitive case):

\[ B_1 = p_1 \rightarrow \square \diamond p_1, \quad B_{i+1} = p_{i+1} \rightarrow \square (\diamond p_{i+1} \lor B_i) \]

Proposition

\( B_h \) is valid in a transitive \( F \) iff the height of \( F \) \( \leq h \).

A logic \( L \supseteq K4 \) is of finite height if it contains \( B_h \) for some \( h \).

Theorem (Segerberg, 1971; Maksimova, 1975)

For a logic \( L \supseteq K4 \),

\( L \) is locally tabular iff it is of finite height.
A logic \( L \) is pretransitive if the master modality is expressible in \( L \). Formally:

A logic \( L \) is said to be \textit{pretransitive} (or \textit{conically expressive}), if there exists a formula \( \chi(p) \) with a single variable \( p \) such that for every Kripke model \( M \) with \( M \models L \) and for every \( w \) in \( M \) we have:

\[
M, w \models \chi(p) \iff \forall u (w R^* u \Rightarrow M, u \models p).
\]

\( L \) is \textit{\( m \)-transitive} iff \( L \vdash \Diamond^{m+1} p \rightarrow \Diamond^{\leq m} p \) for some \( m \geq 0 \), where

\[
\Diamond^0 \varphi := \varphi, \quad \Diamond^{i+1} \varphi := \Diamond \Diamond^i \varphi, \quad \Diamond^{\leq m} \varphi := \bigvee_{i=0}^m \Diamond^i \varphi, \quad \Box^{\leq m} \varphi := \neg \Diamond^{\leq m} \neg \varphi.
\]

**Theorem (Shehtman, 2009)**

- \( L \) is pretransitive iff it is \( m \)-transitive for some \( m \geq 0 \).
- For an \( m \)-transitive logic \( L \), the set \( \{ \varphi \mid L \vdash \varphi^{[m]} \} \) is a logic containing \( S4 \).

Here \( \varphi^{[m]} \) denotes the formula obtained from \( \varphi \) by replacing \( \Diamond \) with \( \Diamond^{\leq m} \) and \( \Box \) with \( \Box^{\leq m} \).
Unimodal case. Necessary condition.

**Theorem (Shehtman, Sh, 2016)**

*Every locally tabular logic is a pretransitive logic of finite height:*

\[ \text{L is locally tabular } \Rightarrow \text{L contains } (\lozenge^{m+1} p \rightarrow \lozenge^{\leq m} p) \land B^{[m]}_h \text{ for some } m, h. \]

Formulas of finite height *(transitive case)*:

\[ B_1 = p_1 \rightarrow \Box \lozenge p_1, \quad B_{i+1} = p_{i+1} \rightarrow \Box (\lozenge p_{i+1} \lor B_i) \]

*Formulas of finite height* *(pretransitive case)*:

\[ B^{[m]}_h \text{ is obtained from } B_h \text{ by replacing } \lozenge \text{ with } \lozenge^{\leq m} \text{ and } \Box \text{ with } \Box^{\leq m}. \]

\[ R^{\leq m} = \bigcup_{0 \leq i \leq m} R^i, \text{ where } R^0 = \text{Id}(W), \quad R^{i+1} = R \circ R^i. \]

\[ R \text{ is } m\text{-transitive}, \text{ if } R^{\leq m} = R^*, \text{ or equivalently, } R^{m+1} \subseteq R^{\leq m}. \]

**Proposition.** \( R \text{ is } m\text{-transitive iff } (W, R) \models \lozenge^{m+1} p \rightarrow \lozenge^{\leq m} p. \)

**Proposition.** For an \( m\)-transitive frame \( F \), \( F \models B^{[m]}_h \iff \text{ht}(F) \leq h. \)
Unimodal case. Necessary condition.

L is locally tabular $\Rightarrow$ L contains $(\Diamond^{m+1} p \rightarrow \Diamond^{\leq m} p) \land B_h^{[m]}$ for some $m, h$.

Every locally tabular logic is a pretransitive logic of finite height, but the converse is not true in general.
Unimodal case. Necessary condition.

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Every locally tabular logic is a pretransitive logic of finite height, but the converse is not true in general.

**Theorem (Kudinov, Sh, 2015)** All the logics \(K + (\Diamond^{m+1} p \rightarrow \Diamond^{\leq m} p) \land B_h^{[m]}\) have the FMP.
Unimodal case. Necessary condition.

L is locally tabular \( \Rightarrow \) L contains \((\Diamond^{m+1} p \rightarrow \Diamond^{\leq m} p) \land B^m_h\) for some \(m, h\).

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**Theorem (Miyazaki, 2004) (Kostrzycka, 2008)** For \(m \geq 2\), all the above logics have Kripke incomplete extensions.

**Corollary** For \(m \geq 2\), none of them are locally tabular.
Unimodal case. Necessary condition.

L is locally tabular \( \Rightarrow \) L contains \((\Box^{m+1} p \rightarrow \Box^{\leq m} p) \land B_h^m\) for some \(m, h\).

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**Corollary** For \(m \geq 2\), none of them are locally tabular.

**Problem, 1960s**

For \(m > 1\), pretransitive logics are very complex and not well-studied. In particular, the FMP (and even the decidability) of the logics \(K + \Box^{m+1} p \rightarrow \Box^{\leq m} p\) is unknown for \(m > 1\).

(“Perhaps one of the most intriguing open problems in Modal Logic” [Wolter F., Zakharyaschev M. Modal decision problems // Handbook of Modal Logic. 2007].)
A partition $\mathcal{A}$ of $F = (W, R)$ is $R$-tuned, if for any $U, V \in \mathcal{A}$

$$\exists u \in U \exists v \in V \: uRv \Rightarrow \forall u \in U \: \exists v \in V \: uRv.$$ 

**Proposition** $\mathcal{A}$ is $R$-tuned iff $\{\bigcup x \mid x \subseteq \mathcal{A}\}$ forms a subalgebra of the modal algebra of $(W, R)$.
A partition $\mathcal{A}$ of $F = (W, R)$ is $R$-tuned, if for any $U, V \in \mathcal{A}$

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Proposition $\mathcal{A}$ is $R$-tuned iff $\{ \bigcup x \mid x \subseteq \mathcal{A} \}$ forms a subalgebra of the modal algebra of $(W, R)$.

Proposition (Franzen, Fine, 1970s)

If for every finite partition $\mathcal{A}$ of $W$ there exists a finite $R$-tuned refinement $\mathcal{B}$ of $\mathcal{A}$, then $\text{Log}(W, R)$ has the FMP.

$\text{Log}(\mathbb{N}, \leq)$ has the FMP: Refine $\mathcal{A}$ in such a way that all elements of $\mathcal{B}$ are infinite or singletons, and singletons cover an initial segment of $\mathbb{N}$.
A partition $\mathcal{A}$ of $F = (W, R)$ is $R$-tuned, if for any $U, V \in \mathcal{A}$

$$\exists u \in U \exists v \in V \ u R v \ \Rightarrow \ \forall u \in U \exists v \in V \ u R v.$$
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$$\exists u \in U \exists v \in V \ u R v \ \Rightarrow \ \forall u \in U \ \exists v \in V \ u R v.$$ 

A class of frames $\mathcal{F}$ is ripe, if there exists $f : \mathbb{N} \rightarrow \mathbb{N}$ s.t. for every finite partition $\mathcal{A}$ of a frame $(W, R) \in \mathcal{F}$ there exists an $R$-tuned refinement $\mathcal{B}$ of $\mathcal{A}$ such that $|\mathcal{B}| \leq f(|\mathcal{A}|)$.

For a class $\mathcal{F}$ of frames let $\text{cl}\mathcal{F}$ be the class of clusters occurring in frames from $\mathcal{F}$: $\text{cl}\mathcal{F} = \{F \mid C \mid F \in \mathcal{F} \text{ and } C \text{ is a cluster in } F\}$.

A class $\mathcal{F}$ of frames is of finite height if $\exists h \in \mathbb{N}$ s.t. $ht(F) \leq h$ for all $F \in \mathcal{F}$.

Theorem (Shehtman, Sh, 2016)

$Log \mathcal{F}$ is locally tabular iff $\mathcal{F}$ is ripe iff $\mathcal{F}$ is of finite height and $\text{cl}\mathcal{F}$ is ripe.
A partition $\mathcal{A}$ of $F = (W, R)$ is $R$-tuned, if for any $U, V \in \mathcal{A}$

$$\exists u \in U \exists v \in V \ uRv \Rightarrow \forall u \in U \exists v \in V \ uRv.$$ 

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For a class $\mathcal{F}$ of frames let $cl\mathcal{F}$ be the class of clusters occurring in frames from $\mathcal{F}$: $cl\mathcal{F} = \{F|C \mid F \in \mathcal{F} \text{ and } C \text{ is a cluster in } F\}$.

A class $\mathcal{F}$ of frames is of finite height if $\exists h \in \mathbb{N}$ s.t. $ht(F) \leq h$ for all $F \in \mathcal{F}$.

**Theorem (Shehtman, Sh, 2016)**

$Log\ F$ is locally tabular iff $\mathcal{F}$ is ripe iff $\mathcal{F}$ is of finite height and $cl\mathcal{F}$ is ripe.

**Theorem (Segerberg, Maksimova, 1970s).** For a logic $L \supseteq K4$, $L$ is locally tabular iff it is of finite height.

**Proof.** If $(W, R)$ is a cluster in a transitive frame, than any partition of $(W, R)$ is $R$-tuned.
Unimodal case. Semantic criterions.

A partition \( A \) of \( F = (W, R) \) is \( R \)-tuned, if for any \( U, V \in A \)

\[ \exists u \in U \exists v \in V \ uRv \ \Rightarrow \ \forall u \in U \exists v \in V \ uRv. \]

A class of frames \( F \) is ripe, if there exists \( f : \mathbb{N} \rightarrow \mathbb{N} \) s.t. for every finite partition \( A \) of a frame \( (W, R) \in F \) there exists an \( R \)-tuned refinement \( B \) of \( A \) such that \( |B| \leq f(|A|) \).

For a class \( F \) of frames let \( \text{cl}F \) be the class of clusters occurring in frames from \( F \): \( \text{cl}F = \{F\upharpoonright C \mid F \in F \text{ and } C \text{ is a cluster in } F\} \).

A class \( F \) of frames is of finite height if \( \exists h \in \mathbb{N} \text{ s.t. } ht(F) \leq h \) for all \( F \in F \).

**Theorem (Shehtman, Sh, 2016)**

\( \text{Log } F \) is locally tabular iff \( F \) is ripe iff \( F \) is of finite height and \( \text{cl}F \) is ripe.

**Theorem.** For a logic \( L \supseteq \text{wK4} = K + \lozenge \lozenge p \rightarrow \lozenge p \lor p \),
\( L \) is locally tabular iff it is of finite height.

**Proof** If \( (W, R) \) is a cluster in a \( \text{wK4} \)-frame, than any partition of \((W, R)\) is \( R \)-tuned.
A partition $A$ of $F = (W, R)$ is $R$-tuned, if for any $U, V \in A$
\[\exists u \in U \exists v \in V \ uRv \ \Rightarrow \ \forall u \in U \ \exists v \in V \ uRv.\]

A class of frames $\mathcal{F}$ is ripe, if there exists $f : \mathbb{N} \rightarrow \mathbb{N}$ s.t. for every finite partition $A$ of a frame $(W, R) \in \mathcal{F}$ there exists an $R$-tuned refinement $B$ of $A$ such that $|B| \leq f(|A|)$.

For a class $\mathcal{F}$ of frames let $\text{cl}\mathcal{F}$ be the class of clusters occurring in frames from $\mathcal{F}$: $\text{cl}\mathcal{F} = \{F|C \mid F \in \mathcal{F} \text{ and } C \text{ is a cluster in } F\}$.

A class $\mathcal{F}$ of frames is of finite height if $\exists h \in \mathbb{N}$ s.t. $ht(F) \leq h$ for all $F \in \mathcal{F}$.

Theorem (Shehtman, Sh, 2016)
$\text{Log } \mathcal{F}$ is locally tabular iff $\mathcal{F}$ is ripe iff $\mathcal{F}$ is of finite height and $\text{cl}\mathcal{F}$ is ripe.

Theorem. For $m > 0$ and a logic $L$ containing $\Diamond^{m+1}p \rightarrow \Diamond p \lor p$,
$L$ is locally tabular iff it is of finite height.
Proof The class of clusters in frames validating $\Diamond^{m+1}p \rightarrow \Diamond p \lor p$ is ripe.
Fix some $n > 0$ and the $n$-modal language with the modalities $\diamondsuit_0, \ldots, \diamondsuit_{n-1}$.

For a Kripke frame $F = (W, (R_i)_{i < n})$, put

$$R_F = \bigcup_{i < n} R_i.$$  

$\sim_F$ is the equivalence relation $R_F^* \cap R_F^* \mathbin{-1}$, where $R_F^*$ is the transitive reflexive closure of $R_F$.

A *cluster* in $F$ is an equivalence class under $\sim_F$.

The *height* of $F$ is the height of $(W, R_F)$.

$F$ is *$m$-transitive* if $R_F$ is $m$-transitive.
Theorem

Every locally tabular $n$-modal logic is a pretransitive logic of finite height:
$L$ is locally tabular $\Rightarrow L$ contains $(\Diamond^{m+1}p \to \Diamond^{\leq m}p) \land B_h^{[m]}$ for some $m, h$.

Master modality (polymodal case)

$\Diamond^{\leq m}\varphi = \lor_{i\leq m} \Diamond^i \varphi$, $\Box^{\leq m}\varphi = \neg \Diamond^{\leq m} \neg \varphi$, where

$\Diamond \varphi = \Diamond^0 \varphi \lor \ldots \lor \Diamond^{n-1} \varphi$

Formulas of finite height (polymodal case):

$B_1^{[m]} = p_1 \to \Box^{\leq m} \Diamond^{\leq m} p_1$, $B_{i+1}^{[m]} = p_{i+1} \to \Box^{\leq m}(\Diamond^{\leq m} p_{i+1} \lor B_i^{[m]})$.

Proposition. $F$ is $m$-transitive iff $F \models \Diamond^{m+1}p \to \Diamond^{\leq m}p$.

Proposition. For an $m$-transitive frame $F$, $F \models B_h^{[m]} \iff ht(F) \leq h$. 
Let \( F = (W, (R_i)_{i < n}) \) be a Kripke frame. A partition \( \mathcal{A} \) of \( W \) is \( F\text{-tuned} \), if for every \( U, V \in \mathcal{A} \), and every \( i < n \)

\[
\exists u \in U \exists v \in V \ uR_i v \quad \Rightarrow \quad \forall u \in U \exists v \in V \ uR_i v.
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Let $F = (W, (R_i)_{i<n})$ be a Kripke frame. A partition $\mathcal{A}$ of $W$ is \textit{F-tuned}, if for every $U, V \in \mathcal{A}$, and every $i < n$

$$\exists u \in U \exists v \in V \ u R_i v \quad \Rightarrow \quad \forall u \in U \ \exists v \in V \ u R_i v.$$ 

A class of frames $\mathcal{F}$ is \textit{ripe}, if there exists $f : \mathbb{N} \to \mathbb{N}$ s.t. for every finite partition $\mathcal{A}$ of a frame $F \in \mathcal{F}$ there exists an $F$-tuned refinement $B$ of $\mathcal{A}$ such that $|B| \leq f(|\mathcal{A}|)$. 

**Theorem (Main result)**

$\log F$ is locally tabular iff $F$ is ripe iff $F$ is of finite height and $\text{cl} F$ is ripe.

**Example**

$K_n$ is the least $n$-modal logic, $\Box \phi = \Box 0 \phi \land ... \land \Box n-1 \phi$. 

**Theorem (Gabbay, Shehtman, 1998; Shehtman, 2014).**

For all $h \geq 0$, $K_n + \Box h \perp$ is locally tabular.

**Proof.** The logic is Kripke complete, the height of its frames $\leq h + 1$, their clusters are singletons. Q.E.D.
Let $F = (W, (R_i)_{i<n})$ be a Kripke frame. A partition $\mathcal{A}$ of $W$ is $F$-tuned, if for every $U, V \in \mathcal{A}$, and every $i < n$

$$\exists u \in U \exists v \in V \ u R_i v \Rightarrow \forall u \in U \exists v \in V \ u R_i v.$$ 

A class of frames $\mathcal{F}$ is ripe, if there exists $f : \mathbb{N} \to \mathbb{N}$ s.t. for every finite partition $\mathcal{A}$ of a frame $F \in \mathcal{F}$ there exists an $F$-tuned refinement $B$ of $\mathcal{A}$ such that $|B| \leq f(|\mathcal{A}|)$.

**Theorem (Main result)**

Log $\mathcal{F}$ is locally tabular iff $\mathcal{F}$ is ripe iff $\mathcal{F}$ is of finite height and $\text{cl}\mathcal{F}$ is ripe.

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**Theorem (Gabbay, Shehtman, 1998; Shehtman, 2014).**

For all $h \geq 0$, $K_n + \Box^h \bot$ is locally tabular.
Polymodal case. Semantic criterions.

Let $F = (W, (R_i)_{i < n})$ be a Kripke frame. A partition $\mathcal{A}$ of $W$ is $F$-tuned, if for every $U, V \in \mathcal{A}$, and every $i < n$

$$\exists u \in U \exists v \in V \ u R_i v \ \Rightarrow \ \forall u \in U \ \exists v \in V \ u R_i v.$$ 

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**Proof.**
Polymodal case. Semantic criterions.

Let \( F = (W, (R_i)_{i<n}) \) be a Kripke frame. A partition \( \mathcal{A} \) of \( W \) is \( F\)-tuned, if for every \( U, V \in \mathcal{A} \), and every \( i < n \)

\[
\exists u \in U \exists v \in V \ uR_iv \Rightarrow \forall u \in U \exists v \in V \ uR_iv.
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A class of frames \( \mathcal{F} \) is ripe, if there exists \( f : \mathbb{N} \rightarrow \mathbb{N} \) s.t. for every finite partition \( \mathcal{A} \) of a frame \( F \in \mathcal{F} \) there exists an \( F \)-tuned refinement \( \mathcal{B} \) of \( \mathcal{A} \) such that \( |\mathcal{B}| \leq f(|\mathcal{A}|) \).

**Theorem (Main result)**

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Let $F = (W, (R_i)_{i < n})$ be a Kripke frame. A partition $\mathcal{A}$ of $W$ is $F$-tuned, if for every $U, V \in \mathcal{A}$, and every $i < n$

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**Theorem (Main result)**

Log $\mathcal{F}$ is locally tabular iff $\mathcal{F}$ is ripe iff $\mathcal{F}$ is of finite height and $cl\mathcal{F}$ is ripe.

**Example**

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**Theorem (Gabbay, Shehtman, 1998; Shehtman, 2014).**

For all $h \geq 0$, $K_n + \square^h \bot$ is locally tabular.

**Proof.** The logic is Kripke complete, the height of its frames $\leq h + 1$. 

Let $F = (W, (R_i)_{i < n})$ be a Kripke frame. A partition $A$ of $W$ is $F$-tuned, if for every $U, V \in A$, and every $i < n$

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A class of frames $\mathcal{F}$ is ripe, if there exists $f : \mathbb{N} \to \mathbb{N}$ s.t. for every finite partition $A$ of a frame $F \in \mathcal{F}$ there exists an $F$-tuned refinement $B$ of $A$ such that $|B| \leq f(|A|)$.

Theorem (Main result)

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For all $h \geq 0$, $K_n + \square^h \perp$ is locally tabular.

Proof. The logic is Kripke complete, the height of its frames $\leq h + 1$, their clusters are singletons.
Polymodal case. Semantic criterions.

Let $F = (W, (R_i)_{i<n})$ be a Kripke frame. A partition $\mathcal{A}$ of $W$ is $F$-tuned, if for every $U, V \in \mathcal{A}$, and every $i < n$

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For all $h \geq 0$, $K_n + \square^h \bot$ is locally tabular.

Proof. The logic is Kripke complete, the height of its frames $\leq h + 1$, their clusters are singletons. Q.E.D.
Corollaries. Adding universal modality.

For $F = (W, (R_i)_{i < n})$, let $F_u = (W, (R_i)_{i < n}, W \times W)$. For a class $\mathcal{F}$ of frames put $\mathcal{F}_u = \{F_u | F \in \mathcal{F}\}$.

For an $n$-modal logic $L$, let $L_u$ be the $(n + 1)$-modal logic in the language with the extra modality $[u]$, that is the extension of $L$ with the $S5$-axioms for $[u]$ and the axioms $[u]p \to \Box_ip$ for all $i < n$.

**Proposition.** $F \models L$ iff $F_u \models L_u$.

**Proposition.** If $\mathcal{F}$ is the class of all the frames of $L$ and $L_u$ is Kripke complete, then $L_u = \text{Log } \mathcal{F}_u$.

Extending a logic with $[u]$ is not safe: we can lose the FMP (Wolter, 1994), decidability (Spaan, 1993), Kripke completeness (Kracht, 1999).
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**Theorem**

*If $L$ is locally tabular, then $L_u$ is locally tabular.*

**Proof.**

Trivially, if a partition is tuned for a frame $(W, (R_i)_{i < n})$, then it is tuned for the frame $(W, (R_i)_{i < n}, W \times W)$.
Corollaries. Tense counterpart.

For $F = (W, (R_i)_{i < n})$, put $F_t = (W, (R_i)_{i < n}, (R_i^{-1})_{i < n})$.

For an $n$-modal logic $L$, let $L_t$ be a $2n$-modal logic, which is the extension of $L$ with the axioms $p \rightarrow □_i \Diamond_i^{-1} p$ and $p \rightarrow □_i^{-1} \Diamond_i p$ for all $i < n$.

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**Proposition.** If $F$ is the class of all the frames of $L$ and $L_t$ is Kripke complete, then $L_t = \text{Log} \ F_t$.

Adding the converse modalities is not safe: we can lose the FMP, decidability, Kripke completeness (Wolter, 1995, 1996),...
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Tuned partitions allows us to construct *(minimal) filtrations*. Thus, locally tabular logics *admit filtration*.

**Theorem (Kikot, Zolin, Sh, 2014).** If $L$ admits filtration and $L_t$ is Kripke complete, then $L_t$ has the FMP.

**Theorem**

*If $L$ is locally tabular, then $L_t$ has the finite model property.*
Locally tabular products of modal logics

**Theorem (N. Bezhanishvili, 2002)** Every proper extension of $S5 \times S5$ is locally tabular.
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Problem

*Which properties of $L_1$ and $L_2$ guarantee the local tabularity of $L_1 \times L_2$?*

*What are the properties of $L_1 \times L_2$, if $L_1$ and $L_2$ are locally tabular?*
**Corollaries and problems**

*Log* $\mathcal{F}$ is locally tabular iff $\mathcal{F}$ is ripe iff $\mathcal{F}$ is of finite height and $\text{cl} \mathcal{F}$ is ripe.

**Corollary.** Suppose $L_0$ is a (weakly) canonical pretransitive logic and the class of clusters occurring in its frames is ripe. Then for any logic $L \supseteq L_0$,

$L$ is locally tabular iff $L$ is of finite height.

**Problem.** A syntactic criterion of LT for all modal logics.
Corollaries and problems

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**Corollary.** \( L \) is locally tabular iff \( L \) is the logic of some class \( F \) such that \( F \) is of finite height and \( \text{Log cl } F \) is locally tabular.

In the above, \( \text{Log cl } F \) is a pretransitive logic of height 1 (the master modality satisfies S5).

**Theorem (Kowalski, 2006).** \( L \) is a pretransitive logic of height 1 iff \( \text{Alg } L \) is a discriminator variety.

**Problem.** To describe locally finite modal discriminator varieties.
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\textit{Problem.} To describe locally finite modal discriminator varieties.

\textit{Question.} What is the finite height algebraically?
Concluding remarks

**Theorem (Maksimova, 1975).**

A logic $L \supseteq K4$ is locally tabular iff its 1-generated free algebra $A_L(1)$ is finite.

**Corollary**

Suppose $L_0$ is a canonical pretransitive logic and the class of clusters occurring in its frames is ripe. Then for any logic $L \supseteq L_0$, $L$ is locally tabular iff $A_L(1)$ is finite.

**Proof.**

Recall that in this case $L$ is locally tabular iff $L$ is of finite height. The $\Box^*$-fragment of $L$ is a logic $L^*$ containing $S4$. If $A_L(1)$ is finite, then $A_{L^*}(1)$ is finite, thus, $L^*$ is locally tabular, thus, it is of finite height. Thus, $L$ is of finite height.

**Problem (1970s).**

Does this equivalence hold for every modal logic?

**Question.** Suppose that we know how to tune all two-element partitions of a cluster. Can we tune all finite ones?
Theorem (Maksimova, 1975).

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Suppose \( L_0 \) is a canonical pretransitive logic and the class of clusters occurring in its frames is ripe. Then for any logic \( L \supseteq L_0 \),

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L \text{ is locally tabular iff } \mathcal{A}_L(1) \text{ is finite.}
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Proof.

Recall that in this case \( L \) is locally tabular iff \( L \) is of finite height.

The \( \square^* \)-fragment of \( L \) is a logic \( L_\ast \) containing \( S4 \).

If \( \mathcal{A}_L(1) \) is finite, then \( \mathcal{A}_{L_\ast}(1) \) is finite, thus, \( L_\ast \) is locally tabular, thus, it is of finite height. Thus, \( L \) is of finite height. \( \square \)

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Suppose \( L_0 \) is a canonical pretransitive logic and the class of clusters occurring in its frames is ripe. Then for any logic \( L \supseteq L_0 \),
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Recall that in this case \( L \) is locally tabular iff \( L \) is of finite height.
The \( \square^* \)-fragment of \( L \) is a logic \( L_* \) containing \( S4 \).
If \( \mathcal{A}_L(1) \) is finite, then \( \mathcal{A}_{L_*}(1) \) is finite, thus, \( L_* \) is locally tabular, thus, it is of finite height. Thus, \( L \) is of finite height. \( \square \)

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Does this equivalence hold for every modal logic?

Question. Suppose that we know how to tune all two-element partitions of a cluster. Can we tune all finite ones?
Thank you!