Existentially Closed Brouwerian Semilattices

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Brouwerian semilattices

Definition

Brouwerian semilattices (also called implicative semilattices) are $\wedge$-semilattices with a top element 1 and an implication operation $\to$ satisfying

$$a \wedge b \leq c \qquad \text{iff} \qquad a \leq b \to c$$

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The order is given by $a \leq b$ iff $a \wedge b = a$.

Amalgamation property

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For any ϕ, ψ propositional formulas in said fragment there exists a formula θ of the fragment containing only proposition letters common to ϕ and ψ such that $\phi \to \theta$ and $\theta \to \psi$ are validities.

Locally finiteness

The variety is also *locally finite*: any finitely generated Brouwerian semilattice is actually finite.

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The cardinalities of the finitely generated free Brouwerian semilattices, although finite, grow very rapidly.
It is known that

$$\begin{aligned} \#F_0 &= 1 \\ \#F_1 &= 2 \\ \#F_2 &= 18 \\ \#F_3 &= 623,662,965,552,330 \end{aligned}$$

The size of F_4 is still unknown. It was proved by P. Köhler that the number of meet-irreducible elements of F_4 is $2,494,651,862,209,437$.

Existentially closed Brouwerian semilattices

Definition

A Brouwerian semilattice L is said to be *existentially closed* if for any extension $L \subseteq L'$ and for any existential sentence ϕ in the language of Brouwerian semilattices extended with the names of the elements of L we have that if ϕ is true in L' then it is also true in L.

Existentially closed Brouwerian semilattices

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The existence of a model completion of the theory of Brouwerian semilattices is guaranteed by the following result due to W. Wheeler

Theorem (Wheeler, 1976)

Let $\mathcal{V}$ be an amalgamable and locally finite variety of algebras of a signature having at least one constant symbol. Then the theory of $\mathcal{V}$ admits a model completion.

Therefore the class of the existentially closed Brouwerian semilattices is elementary.

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Therefore the class of the existentially closed Brouwerian semilattices is elementary.

It is thus natural to look for an axiomatization of said model completion.

Axiomatization

Note that supplying an axiomatization of the model completion for this kind of algebraic theories is usually a hard task. An axiomatization for the model completion of Heyting algebras is still unknown.

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It is well-known that the existentially closed Boolean algebras are exactly the atomless ones.

Axiomatization

We have proven that the following three axioms together with the axioms of Brouwerian semilattices give a finite axiomatization of the model completion of Brouwerian semilattices.

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We use the abbreviation $a \ll b$ for $a \leq b$ and $b \rightarrow a = a$.

[Density 1] For every c there exists an element b different from 1 such that $b \ll c$.

[Density 2] For every c, a_1, a_2, d such that $a_1, a_2 \neq 1$, $a_1 \ll c$, $a_2 \ll c$ and $d \rightarrow a_1 = a_1$, $d \rightarrow a_2 = a_2$ there exists an element b different from 1 such that:

$$\begin{aligned} a_1 &\ll b \\ a_2 &\ll b \\ b &\ll c \\ d \rightarrow b &= b \end{aligned}$$

Axiomatization

[Splitting] For every a, b_1, b_2 such that $1 \neq a \ll b_1 \wedge b_2$ there exist elements a_1 and a_2 different from 1 such that:

$$b_1 \geq a_1 = a_2 \to a$$
$$b_2 \geq a_2 = a_1 \to a$$
$$a_2 \to b_1 = b_2 \to b_1$$
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Some properties of ex. closed Brouwerian semilattices

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- the join of any pair of incomparable elements does not exist
- there are no meet-irreducible elements

Proving the result

The remaining of this talk will be devoted to explaining how we proved this result.

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Thanks to the locally finiteness and the amalgamability, by an easy model-theoretic reasoning we obtain:

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We thus want to study finite extensions of finite Brouwerian semilattices.

Finite duality

Any finite Brouwerian semilattice is complete and thus it is a lattice.
It is also distributive, because it exists for any element a the right adjoint of $a \wedge -$ given by $a \to -$.

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So, we expect the existence of a duality between finite Brouwerian semilattices and finite posets.

Finite duality

This is indeed the case. Before giving the full description of the finite duality due to P. Köhler, we shall take a change of perspective.

co-Brouwerian semilattices

A *co-Brouwerian semilattice*, CBS for short, is a structure obtained by reversing the order of a Brouwerian semilattice.
We will work with CBSes instead of Brouwerian semilattices.

There are two reasons for this decision: it will make the finite duality easier to work with and it will help to understand intuitively the constructions featured in the proofs.

co-Brouwerian semilattices

Therefore CBSes are $\vee$-semilattices with a minimum element 0 and a difference operation with the property

$$a - b \leq c \qquad \text{iff} \qquad a \leq b \vee c$$

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Clearly any result concerning Brouwerian semilattices can be translated in the language of CBSes by reversing the order: 1 is replaced by 0, meets are replaced by joins and $a \to b$ is replaced by $b - a$.

Finite duality

Theorem (Köhler, 1981)

The category of finite CBSes is dually equivalent to the category $\mathbf{P}$ *whose objects are finite posets and whose morphisms are partial maps* $f : P \to Q$ *with the following properties:*

- *(strict order preserving) for any* $a, b \in \mathit{dom}\ f$ *if* $a < b$ *then* $f(a) < f(b)$
- *for any* $p \in \mathit{dom}\ f$, $q \in Q$ *if* $f(p) < q$ *then there exists* $p' \in P$ *such that* $p < p'$ *and* $f(p') = q$.

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On the other hand, to any finite poset P it is associated the CBS $\mathcal{D}(P)$ given by the downward closed subsets of P.
The join operation is the set-theoretic union of downsets, the zero element is the empty downset, the difference of two downsets A, B is given by $A - B = \downarrow(A \setminus B)$.

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The join operation is the set-theoretic union of downsets, the zero element is the empty downset, the difference of two downsets A, B is given by $A - B = \downarrow(A \setminus B)$.

Furthermore, to any **P**-morphism between finite posets $f : P \to Q$ it is associated the morphism of CBSes $\varphi : \mathcal{D}(Q) \to \mathcal{D}(P)$ given by $\varphi(D) = \downarrow f^{-1}(D)$ for any $D \in \mathcal{D}(Q)$.

Finite duality

It turns out that:

Proposition

Quotients of finite CBSes correspond to total and injective **P***-morphisms.*
Embeddings of finite CBSes correspond to surjective **P***-morphisms.*

Minimal extensions of finite CBSes

Definition

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The following proposition is very useful:

Proposition

The minimal finite extensions of CBSes are exactly the ones dual to surjective $\mathbf{P}$*-morphisms* $f : P \to Q$ *such that* $\#P = \#Q + 1$*. (we call these morphisms minimal)*

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First kind: the ones in which $\#\mathrm{dom}\ f = \#P - 1 = \#Q$, i.e. there is exactly one element of P outside the domain and the map is an isomorphism when restricted on its domain.

Second kind: the ones in which f is total, note that in this case all of its fibers contain exactly one element except one fiber having two elements.

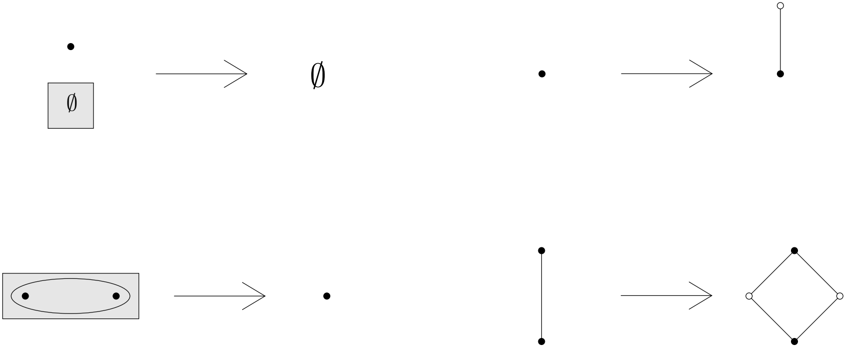
Minimal finite extensions of CBSes

As a result, there are two kinds of minimal finite extensions of finite CBSes:

The first kind is given by the ones preserving the join-irridducibility of all the elements.

The ones of the second kind preserve the join-irridducibility of all the elements except one which becomes the join of two new join-irreducible elements.

Examples



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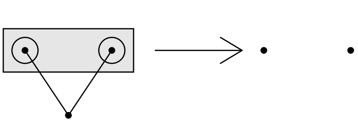

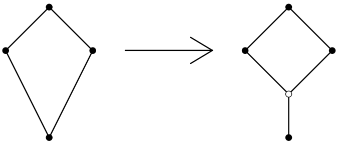

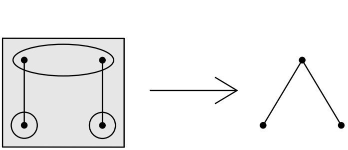

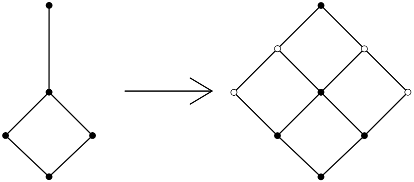

Minimal finite extensions

Since, as we noted before, any finite extension of CBSes is a composition of finite minimal ones, we can replace finite extensions with minimal finite extensions in the characterization of the existentially closed structures we have given before.

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As a consequence of this, we are interested in studying all the possible minimal finite extensions of any finite CBS.

Footprints of minimal finite extensions

It turns out that, given a finite poset Q, the minimal surjective **P**-morphisms with codomain Q correspond up to isomorphism to some 'footprints' inside Q.

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The ones of the first kind correspond to couples (D, U) where D, U are respectively a downset and an upset of Q such that $D \cap U = \emptyset$ and for any $d \in D$ and $u \in U$ we have $d \leq u$.

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The ones of the second kind correspond to triples (g, D_1, D_2) where $g \in Q$ and D_1, D_2 are downsets of Q such that $D_1 \cup D_2 = \downarrow g \setminus \{g\}$.

Existentially closed CBSes

In this way, we can obtain another characterization of the existentially closed CBSes: a CBS L is existentially closed iff for any finite sub-CBS $L_0 \subseteq L$ and for any ‘footprint’ inside L_0 there is minimal finite extension of L_0 inside L having that ‘footprint’.

Density axioms

$a \ll b$ means $a \leq b$ and $b - a = b$.

[Density 1 Axiom] For every c there exists $b \neq 0$ such that $c \ll b$

[Density 2 Axiom] For every c, a_1, a_2, d such that $a_1, a_2 \neq 0$, $c \ll a_1$, $c \ll a_2$ and $a_1 - d = a_1$, $a_2 - d = a_2$ there exists an element b different from 0 such that:

$$\begin{aligned} c &\ll b \\ b &\ll a_1 \\ b &\ll a_2 \\ b - d &= b \end{aligned}$$

Splitting axiom

[Splitting Axiom] For every a, b_1, b_2 such that $b_1 \vee b_2 \ll a \neq 0$ there exist elements a_1 and a_2 different from 0 such that:

$$a - a_1 = a_2 \geq b_2$$
$$a - a_2 = a_1 \geq b_1$$
$$b_2 - a_1 = b_2 - b_1$$
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Lemma

Let $\theta(\underline{x})$ and $\phi(\underline{x}, \underline{y})$ be quantifier-free formulas in the language of CBSes. Assume that for every finite CBS L_0 and every tuple $\underline{a}$ of elements of L_0 such that $L_0 \vDash \theta(\underline{a})$, there exists an extension L_1 of L_0 which satisfies $\exists \underline{y} \phi(\underline{a}, \underline{y})$.
Then every existentially closed CBS satisfies the following sentence:

$$\forall \underline{x}(\theta(\underline{x}) \longrightarrow \exists \underline{y} \phi(\underline{x}, \underline{y}))$$

In our cases, to construct the required extension L_1 is relatively easy using the finite duality.

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We want to prove that given a CBS L satisfying the Splitting axiom, $L_0 \subseteq L$ a finite sub-CBS, g join-irreducible element of L_0 and $h_1, h_2 \in L_0$ such that $h_1 \vee h_2 \ll g$ there exists $L_0' \subseteq L$ such that $L_0 \subseteq L_0'$ is a minimal extension of the second kind of L_0 corresponding to the 'footprint' (g, h_1, h_2).

Axiomatization

The proof is by induction on a natural number n that is associated to (h_1, h_2) which measures how much smaller is $h_1 \wedge h_2$ (taken inside L_0) than h_1 and h_2.

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In this way we get two elements of L that generate over L_0 precisely an extension L_0' which is a minimal extension of the second kind corresponding to the 'footprint' (g, h_1, h_2).

Thanks for your attention!

The complete investigation can be found in the preliminary manuscript at the following link: `http://arxiv.org/abs/1702.08352`

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