Types and models in core fuzzy predicate logics

Guillermo Badia

Department of Knowledge-Based Mathematical Systems,
Johannes Kepler University (Linz, Austria)
This work is supported by the Austrian Science Fund (FWF): project I 1923-N25 (*New perspectives on residuated posets*).
Joint work with Carles Noguera

27 June 2017

Motivation

Fuzzy relations were famously introduced by Zadeh in 1965 as relations with graded membership (with grades normally in the interval $[0, 1]$). Goguen in 1967 generalized this idea to membership graded by an arbitrary lattice.

Graded notions are pervasive in everyday discourse, hence it is hardly surprising that this formal representation found a wide range of applications, e.g., in the modeling of the degree of certainty of a given medical diagnosis.

Motivation (cont'd)

Systems of such fuzzy relations have been rediscovered in the area of weighted CSP, where the main concern can be described as the problem of computing the value of primitive positive fuzzy formulas. There they consider finite structures (infinite if the interest is on infinite templates) with weighted constraints, which are essentially fuzzy relations. This observation has recently been made by Rostislav Horčik.

Motivation (cont'd)

Fuzzy model theory is *the general study of the construction and classification of systems of fuzzy relations (weighted structures).*

It's got the potential to impact the field of weighted CSP in the same way that traditional model theory has impacted CSP.

The area was essentially started by Petr Hájek and Petr Cintula in their paper "On theories and models in fuzzy predicate logics" (JSL, 2006).

Aim

We will show how to construct systems of fuzzy MTL-chain-relations where many types are realized (saturated models) and where few types are.

The language

Suppose our language L has available some collection of relation and constant symbols. Fuzzy formulas will be built as follows:

$$\varphi ::= R^n t_0, \ldots t_n \mid \bot \mid \varphi \wedge \psi \mid \varphi \to \psi \mid \varphi \cdot \psi \mid \forall \bar{x} \varphi \mid \exists \bar{x} \varphi$$

Moreover, $\varphi \vee \psi := ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$ and $\top := \bot \to \bot$.

MTL-chains

An **MTL**-chain is a structure $\langle A, \wedge, \vee, \cdot, \Rightarrow, 1, 0\rangle$ such that:

- $\langle A, \cdot, 1\rangle$ is a commutative monoid
- $\langle A, \wedge, \vee, 1, 0\rangle$ is a linearly ordered bounded lattice
- $(x \Rightarrow y) \vee (y \Rightarrow x) = 1$
- The residuation law holds: $xy \leqslant z$ iff $y \leqslant x \Rightarrow z$.

Models

For an MTL-chain $\mathbf{B}$, a *structure* will be formed by a collection of $\mathbf{B}$-relations with a fixed base (that is, they all have the same domain), plus perhaps some distinguished elements of such base.

We denote models as structures as pairs $\langle \mathbf{B}, \mathbf{M} \rangle$ and use M to denote the fixed base of the relations.

The truth value of a given formula $\varphi(\bar{x})$ for some sequence of elements $\bar{a}$ from M is specified interpreting the connectives in the obvious way while the quantifiers are suprema ($\exists$) and infima ($\forall$).

Models (cont'd)

We require that our structures are safe: the value of every formula is defined. Then we call the *models*. We write $\langle \mathbf{B}, \mathbf{M} \rangle \models \varphi[\bar{a}]$ if $||\varphi(\bar{a})|| = 1$.

Moreover, we are only interested in 1-witnessed models: $||\exists x \varphi(x)|| = 1$ means that $||\varphi[d]|| = 1$ for some element d of its domain of individuals.

A model is *exhaustive* if every element of the algebra is the value of some formula for some assignment of objects.

Tableaux

A *tableau* is going to be a pair (T, U) such that T and U are theories.

A tableau is *satisfied* by a model $\langle \mathbf{B}, \mathbf{M} \rangle$, if we have that both $\langle \mathbf{B}, \mathbf{M} \rangle \models T$ and, for all $\varphi \in U$, $\langle \mathbf{B}, \mathbf{M} \rangle \not\models \varphi$.

We may define the expression $(T, U) \models \varphi$ as meaning that for any model that satisfies (T, U), the model must make φ true as well.

A tableau (T, U) is said to be *consistent* if $T \vdash \bigvee U_0$ for no finite $U_0 \subseteq U$.

Model existence theorem

Theorem

(Model Existence Theorem) Let (T, U) be a consistent tableau. Then there is a model satisfying (T, U).

Tableaux almost strong S-closedness

A topological space is said to be *strongly S-closed* if every family of open sets with the finite intersection property has a non-empty intersection. Moreover, we will say that a space is *almost strongly S-closed* if every family of *basic* open sets with the finite intersection property has a non-empty intersection. There's a topology such that the name of the result below makes sense.

Corollary

(Tableaux almost strong S-closedness) Let (T, U) be a tableau. If every (T_0, U_0), with $|T_0|, |U_0|$ finite and $T_0 \subseteq T$ and $U_0 \subseteq U$, is satisfiable, then (T, U) is satisfied in some model.

Elementary amalgamation

Let us briefly see an application of tableaux almost strong S-closedness. We will write $\langle \mathbf{B}_1, \mathbf{M}_1 \rangle \Rrightarrow \langle \mathbf{B}_2, \mathbf{M}_2 \rangle$ when every formula taking value 1 in $\langle \mathbf{B}_1, \mathbf{M}_1 \rangle$, takes value 1 in $\langle \mathbf{B}_2, \mathbf{M}_2 \rangle$ as well, i.e., $\mathrm{Th}\langle \mathbf{B}_1, \mathbf{M}_1 \rangle \subseteq \mathrm{Th}\langle \mathbf{B}_2, \mathbf{M}_2 \rangle$.

Elementary amalgamation (cont'd)

Proposition

(Elementary amalgamation) Let $\langle \mathbf{B}_1, \mathbf{M}_1 \rangle$ *and* $\langle \mathbf{B}_2, \mathbf{M}_2 \rangle$ *be two structures. Moreover, suppose that* $\overline{a}$ *is a sequence of elements of* M_1 *and* $\overline{b}$ *a sequence of* M_2 *of corresponding length such that* $(\mathbf{B}_1, \mathbf{M}_1, \overline{a}) \Rrightarrow (\mathbf{B}_2, \mathbf{M}_2, \overline{b})$. *Then there is a structure* $(\mathbf{C}, \mathbf{N})$ *into which* $\langle \mathbf{B}_1, \mathbf{M}_1 \rangle$ *is* L*-elementarily mapped by* (g, f) *while* $\langle \mathbf{B}_2, \mathbf{M}_2 \rangle$ *is* L*-elementarily embedded (taking isomorphic copies, we may assume that* $\langle \mathbf{B}_2, \mathbf{M}_2 \rangle$ *is just an* L*-elementary substructure). Furthermore, we can guarantee that* $f(\overline{a}) = \overline{b}$.

Elementary amalgamation (cont'd)

The situation is described by the following picture:

$$\begin{tikzcd}
& (\mathbf{C}, \mathbf{N}, \bar{b}) & \\
(\mathbf{B}_2, \mathbf{M}_2, \bar{b}) \arrow[ur, dashed, "\preccurlyeq"'] \arrow[rr, dash, "\Lleftarrow_L"'] & & (\mathbf{B}_1, \mathbf{M}_1, \overline{a}) \arrow[ul, dashed, "{(g, f)}"']
\end{tikzcd}$$

Unions of chains

A sequence $\langle\langle \mathbf{B}_i, \mathbf{M}_i\rangle\rangle_{i<\gamma}$ of models is a called a *chain* when for all $i < j < \gamma$ we have that $\langle \mathbf{B}_i, \mathbf{M}_i\rangle$ is a substructure of $\langle \mathbf{B}_j, \mathbf{M}_j\rangle$. If, moreover, these substructures are elementary, we speak of an *elementary chain*.

The *union* of the chain $\langle\langle \mathbf{B}_i, \mathbf{M}_i\rangle\rangle_{i<\gamma}$ is the structure $\langle \mathbf{B}, \mathbf{M}\rangle$ where $\mathbf{B}$ is the classical union model of the classical chain of algebras $(\mathbf{B}_i)_{i<\gamma}$ while $\mathbf{M}$ is defined by taking as its domain $\bigcup_{i<\gamma} M_i$, interpreting the constants of the language as they were interpreted in each $\mathbf{M}_i$ and similarly with the relational symbols of the language.

Unions of chains (cont'd)

Theorem

(Tarski-Vaught theorem on unions of elementary chains) Let $\langle \mathbf{B}, \mathbf{M} \rangle$ be the union of the elementary chain $\langle \langle \mathbf{B}_i, \mathbf{M}_i \rangle \rangle_{i<\gamma}$. Then for every sequence $\overline{a}$ of elements of $\mathbf{M}_i$ and formula φ, $||\varphi[\overline{a}]||^{\langle \mathbf{B}, \mathbf{M} \rangle} = ||\varphi[\overline{a}]||^{\langle \mathbf{B}_i, \mathbf{M}_i \rangle}$. Moreover, the union is exhaustive.

Models realizing many types

Let $\langle \mathbf{B}, \mathbf{M} \rangle$ be a model. If (p, p') is a pair of sets of formulas in some variable x and parameters in some $A \subseteq M$, we will call p a type of $\langle \mathbf{B}, \mathbf{M} \rangle$ in A if the tableau $(Th_A\langle \mathbf{B}, \mathbf{M} \rangle \cup p, \overline{Th}_A\langle \mathbf{B}, \mathbf{M} \rangle \cup p')$ is satisfiable (consistent). We will denote the set of all such types by $S^{\langle \mathbf{B}, \mathbf{M} \rangle}(A)$.

Models realizing many types (cont'd)

A type of the form $(p, \varnothing)$ will be called a *left type*. We might also write it as simply p. Left types are characterizable in the following way.

For any cardinal κ, a model $\langle \mathbf{B}, \mathbf{M} \rangle$ is said to be left κ-saturated if for any $A \subseteq M$ such that $|A| < \kappa$, any left type in $S^{\langle \mathbf{B}, \mathbf{M} \rangle}(A)$ is satisfiable in $\langle \mathbf{B}, \mathbf{M} \rangle$.

Models realizing many types (cont'd)

Theorem

For each cardinal κ, each model can be elementarily extended to a left κ^+-saturated model.

An application

An $\exists_1$-mapping from $\langle \mathbf{B}_1, \mathbf{M}_1 \rangle$ into $\langle \mathbf{B}_2, \mathbf{M}_2 \rangle$ is a pair (g, f) with $f : M_1 \longrightarrow M_2$ and g a homomorphism defined on at least $\{||\varphi(\bar{a})||^{\langle \mathbf{B}, \mathbf{M} \rangle} : \bar{a} \in A^n$ for some n, φ is $\exists_1\}$ such that $g(||\varphi(\bar{a})||^{\langle \mathbf{B}_1, \mathbf{M}_1 \rangle}) = ||\varphi(f(\bar{a}))||^{\langle \mathbf{B}_2, \mathbf{M}_2 \rangle}$. Similarly define a $\forall_1$-mapping.

Theorem

Let φ, χ be formulas and T a theory such that for any two structures $\langle \mathbf{B}_1, \mathbf{M}_1 \rangle$ and $\langle \mathbf{B}_2, \mathbf{M}_2 \rangle$, which are models of T, with a $\exists_1$-mapping (g, f) from the first one to the second one, $\langle \mathbf{B}_1, \mathbf{M}_1 \rangle \models \varphi[\overline{a}]$ only if $\langle \mathbf{B}_2, \mathbf{M}_2 \rangle \models \chi[f\overline{a}]$. Then there is a $\exists_1$-formula ψ such that $T, \varphi \vdash \psi$ and $T, \psi \vdash \chi$.

An application (cont'd)

Corollary

Let φ be a formula and T a theory such that for any two structures $\langle \mathbf{B}_1, \mathbf{M}_1 \rangle$ and $\langle \mathbf{B}_2, \mathbf{M}_2 \rangle$, which are models of T, with a $\exists_1$-mapping (g, f) from the first one to the second one, $\langle \mathbf{B}_1, \mathbf{M}_1 \rangle \models \varphi[\overline{a}]$ only if $\langle \mathbf{B}_2, \mathbf{M}_2 \rangle \models \varphi[f\overline{a}]$ iff there is a $\exists_1$-formula ψ such that $T, \varphi \vdash \psi$ and $T, \psi \vdash \varphi$.

Models realizing few types

A type (p, p') of (T, U) is *unsupported* if for any formulas φ, φ' such that $(T \cup \{\varphi\}, U \cup \{\varphi'\})$ is satisfiable,[1] there are $\psi \in p, \psi' \in p'$ such that $(T \cup \{\varphi\}, U \cup \{\varphi'\}) \not\models \psi$ or $(T \cup \{\varphi, \psi'\}, U \cup \{\varphi'\})$ is satisfiable.

Models realizing few types (cont'd)

Theorem

(Omitting types) Fix a countable language. Let (T, U) be a tableau realized by some model and (p, p') a unsupported n-type of (T, U). Then there is a model satisfying (T, U) which omits (p, p').

Theorem

(Omitting countably many types) Fix a countable language. Let (T, U) be a tableau realized by some model and $(p_i, p_i')(i < \omega)$ a sequence of unsupported n-types of (T, U). Then there is a model satisfying (T, U) which omits $(p_i, p_i')(i < \omega)$.

Proposition

Suppose we have binary symbols in our language $<$ and R. Let $\langle \mathbf{B}, \mathbf{M} \rangle$ be a countable model of the theory (Γ, Δ) where

$$\Gamma = \{\forall x, y(x < y \vee R(x,y) \vee y < x)\} \cup \{\forall x, y, z(R(x,y) \wedge R(y,z) \to R(x,z))\} \cup \{\forall z(\forall x \exists y > x \exists v < z(\psi(v,y)) \to \exists v < z \forall x \exists y > x(\psi(v,y)))\} \cup \{\forall x_0, \dots x_n \exists y(\textstyle\bigwedge_{i \leqslant n} x_i < y) : n < \omega\}$$

and

$$\Delta = \varnothing$$

Then there is an L-elementary extension $(\mathbf{A}, \mathbf{N})$ of $\langle \mathbf{B}, \mathbf{M} \rangle$, such that if $b \in N \backslash M$ is such that $R(b,c)$ does not hold in $(\mathbf{A}, \mathbf{N})$ for any $c \in M$, then, given $a \in M$, $a < b$ must hold in $(\mathbf{A}, \mathbf{N})$ (this model might be called an end extension of $\langle \mathbf{B}, \mathbf{M} \rangle$ relative to R).

The end

Many thanks!