Expansions of Heyting algebras

Christopher Taylor

La Trobe University

Topology, Algebra, and Categories in Logic
Prague, 2017
Motivation

Congruences on Heyting algebras are determined exactly by filters of the underlying lattice – what about algebras with a Heyting algebra reduct?

Boolean algebras with operators. If $B$ is a boolean algebra equipped with finitely many (dual normal) operators, i.e., unary operations $f_1, \ldots, f_n$ satisfying

$$f_i(x \land y) = f_i x \land f_i y,$$

then congruences on $B$ are determined by filters closed under the map $d(x) = f_1 x \land f_2 x \land \ldots \land f_n x$.

Double-Heyting algebras. Double-Heyting algebras have their congruences determined by filters closed under the map $d(x) = (1 \land -x) \to 0$. 

2/16
Motivation

Congruences on Heyting algebras are determined exactly by filters of the underlying lattice – what about algebras with a Heyting algebra reduct?

- **Boolean algebras with operators.** If $B$ is a boolean algebra equipped with finitely many (dual normal) operators, i.e., unary operations $f_1, \ldots, f_n$ satisfying

  $$f_i(x \land y) = f_ix \land f_iy, \quad f_i1 = 1,$$

  ▶ Double-Heyting algebras. Double-Heyting algebras have their congruences determined by filters closed under the map $dx = f_1x \land f_2x \land \ldots \land f_nx$. 

\[2/16\]
Motivation

Congruences on Heyting algebras are determined exactly by filters of the underlying lattice – what about algebras with a Heyting algebra reduct?

- **Boolean algebras with operators.** If $B$ is a boolean algebra equipped with finitely many (dual normal) operators, i.e., unary operations $f_1, \ldots, f_n$ satisfying

$$f_i(x \land y) = f_ix \land f_iy, \quad f_i1 = 1,$$

then congruences on $B$ are determined by filters closed under the map

$$dx = f_1x \land f_2x \land \ldots \land f_nx$$
Motivation

Congruences on Heyting algebras are determined exactly by filters of the underlying lattice – what about algebras with a Heyting algebra reduct?

▶ **Boolean algebras with operators.** If $B$ is a boolean algebra equipped with finitely many (dual normal) operators, i.e., unary operations $f_1, \ldots, f_n$ satisfying

\[ f_i(x \land y) = f_ix \land f_iy, \quad f_i1 = 1, \]

then congruences on $B$ are determined by filters closed under the map

\[ dx = f_1x \land f_2x \land \ldots \land f_nx \]

▶ **Double-Heyting algebras.** Double-Heyting algebras have their congruences determined by filters closed under the map

\[ dx = (1 \div x) \rightarrow 0 \]
Preliminaries

Definition
An algebra $A = \langle A; M, \vee, \wedge, \rightarrow, 0, 1 \rangle$ is an expanded Heyting algebra (EHA) if the reduct $\langle A, \vee, \wedge, \rightarrow, 0, 1 \rangle$ is a Heyting algebra and $M$ is a set of operations on $A$. 

Let $x \leftrightarrow y = x \rightarrow y \wedge y \rightarrow x$. Recall that if $A$ is a Heyting algebra and $F \subseteq A$ is a filter, then the binary relation $\theta(F) = \{ (x, y) \mid x \leftrightarrow y \in F \}$ is a congruence on $A$.

Definition
A filter $F \subseteq A$ is compatible with an $n$-ary operation $f$ on $A$ if $\{ x_i \leftrightarrow y_i \mid i \leq n \} \subseteq F$ implies $f(\vec{x}) \leftrightarrow f(\vec{y}) \in F$.

Theorem
If $A$ is an EHA then $\theta(F)$ is a congruence on $A$ if and only if $F$ is compatible with $f$ for every $f \in M$. 

Preliminaries

Definition
An algebra \( \mathbf{A} = \langle A; M, \lor, \land, \rightarrow, 0, 1 \rangle \) is an \textit{expanded Heyting algebra} (EHA) if the reduct \( \langle A, \lor, \land, \rightarrow, 0, 1 \rangle \) is a Heyting algebra and \( M \) is a set of operations on \( A \).

Let \( x \leftrightarrow y = x \rightarrow y \land y \rightarrow x \). Recall that if \( \mathbf{A} \) is a Heyting algebra and \( F \subseteq A \) is a filter, then the binary relation \( \theta(F) = \{(x, y) \mid x \leftrightarrow y \in F\} \) is a congruence on \( \mathbf{A} \).
Preliminaries

Definition
An algebra $A = \langle A; M, \lor, \land, \rightarrow, 0, 1 \rangle$ is an expanded Heyting algebra (EHA) if the reduct $\langle A, \lor, \land, \rightarrow, 0, 1 \rangle$ is a Heyting algebra and $M$ is a set of operations on $A$.

- Let $x \leftrightarrow y = x \rightarrow y \land y \rightarrow x$. Recall that if $A$ is a Heyting algebra and $F \subseteq A$ is a filter, then the binary relation $	heta(F) = \{(x, y) | x \leftrightarrow y \in F\}$ is a congruence on $A$.

Definition
A filter $F \subseteq A$ is compatible with an $n$-ary operation $f$ on $A$ if

$$\{x_i \leftrightarrow y_i | i \leq n\} \subseteq F \text{ implies } f(\vec{x}) \leftrightarrow f(\vec{y}) \in F.$$
Definition
An algebra $A = \langle A; M, \lor, \land, \rightarrow, 0, 1 \rangle$ is an expanded Heyting algebra (EHA) if the reduct $\langle A, \lor, \land, \rightarrow, 0, 1 \rangle$ is a Heyting algebra and $M$ is a set of operations on $A$.

- Let $x \leftrightarrow y = x \rightarrow y \land y \rightarrow x$. Recall that if $A$ is a Heyting algebra and $F \subseteq A$ is a filter, then the binary relation $\theta(F) = \{(x, y) \mid x \leftrightarrow y \in F\}$ is a congruence on $A$.

Definition
A filter $F \subseteq A$ is compatible with an $n$-ary operation $f$ on $A$ if

$$\{x_i \leftrightarrow y_i \mid i \leq n\} \subseteq F \text{ implies } f(\bar{x}) \leftrightarrow f(\bar{y}) \in F.$$ 

Theorem
If $A$ is an EHA then $\theta(F)$ is a congruence on $A$ if and only if $F$ is compatible with $f$ for every $f \in M$. 

3 / 16
Normal filter terms

Any unquantified $A$ from now on is a fixed but arbitrary EHA.
Normal filter terms

Any unquantified $A$ from now on is a fixed but arbitrary EHA.

Definition
A filter $F$ of $A$ will be called a normal filter if it is compatible with every $f \in M$, or equivalently, if $\theta(F)$ is a congruence on $A$. 
Normal filter terms

Any unquantified $A$ from now on is a fixed but arbitrary EHA.

**Definition**
A filter $F$ of $A$ will be called a *normal filter* if it is compatible with every $f \in M$, or equivalently, if $\theta(F)$ is a congruence on $A$.

**Definition**
Let $t$ be a unary term in the language of $A$. We say that $t$ is a *normal filter term* (on $A$) provided that it is order-preserving, and for every filter $F$ of $A$, the filter $F$ is a normal filter if and only if $F$ is closed under $t^A$. 


Normal filter terms

Any unquantified $A$ from now on is a fixed but arbitrary EHA.

**Definition**
A filter $F$ of $A$ will be called a *normal filter* if it is compatible with every $f \in M$, or equivalently, if $\theta(F)$ is a congruence on $A$.

**Definition**
Let $t$ be a unary term in the language of $A$. We say that $t$ is a *normal filter term* (on $A$) provided that it is order-preserving, and for every filter $F$ of $A$, the filter $F$ is a normal filter if and only if $F$ is closed under $t^A$.

**Example**
The identity function is a normal filter term for unexpanded Heyting algebras.
Normal filter terms

Hence, the algebras from before have normal filter terms.

- **Boolean algebras with operators.** If $B$ is a boolean algebra equipped with unary operators $f_1, \ldots, f_n$, then congruences on $B$ are determined by filters closed under the map

$$dx = f_1 x \wedge f_2 x \wedge \ldots f_n x$$

- **Double-Heyting algebras.** Double-Heyting algebras have their congruences determined by filters closed under the map

$$dx = (1 \div x) \rightarrow 0$$
Normal filter terms

Hence, the algebras from before have normal filter terms.

► **Boolean algebras with operators.** If $\mathbf{B}$ is a boolean algebra equipped with unary operators $f_1, \ldots, f_n$, then congruences on $\mathbf{B}$ are determined by filters closed under the map

$$dx = f_1 x \land f_2 x \land \ldots \land f_n x$$

► **Double-Heyting algebras.** Double-Heyting algebras have their congruences determined by filters closed under the map

$$dx = (1 \div x) \rightarrow 0$$

Let us say that a class of similar algebras has a normal filter term $t$ if $t$ is a normal filter term for each of those algebras.
Constructing normal filter terms

Let $f$ be an $n$-ary operation on $A$. For each $a \in A$, define the set

$$f^{a} = \{ f(\vec{x}) \leftrightarrow f(\vec{y}) \mid (\forall i \leq n) \ x_i, y_i \in A \text{ and } x_i \leftrightarrow y_i \geq a \}.$$
Constrcting normal filter terms

Let $f$ be an $n$-ary operation on $A$. For each $a \in A$, define the set

$$f^{\leftrightarrow}(a) = \{ f(\vec{x}) \leftrightarrow f(\vec{y}) \mid (\forall i \leq n) \ x_i, y_i \in A \text{ and } x_i \leftrightarrow y_i \geq a \}.$$ 

Now define the partial operation $[M]$ by

$$[M]a = \bigwedge \bigcup \{ f^{\leftrightarrow}(a) \mid f \in M \}.$$
Constructing normal filter terms

Let $f$ be an $n$-ary operation on $A$. For each $a \in A$, define the set

$$f^{\leftrightarrow}(a) = \{ f(\vec{x}) \leftrightarrow f(\vec{y}) \mid (\forall i \leq n) \ x_i, y_i \in A \text{ and } x_i \leftrightarrow y_i \geq a \}.$$ 

Now define the partial operation $[M]$ by

$$[M]a = \bigwedge \bigcup \{ f^{\leftrightarrow}(a) \mid f \in M \}.$$ 

If it is defined everywhere then we say that $[M]$ exists in $A$. 

![Diagram](attachment:diagram.png)
Constructing normal filter terms

A unary map $f$ is an operator\(^1\) if $f(x \land y) = fx \land fy$ and $f1 = 1$.

\(^1\)Actually a dual normal operator
Constructing normal filter terms

A unary map $f$ is an operator\(^1\) if $f(x \land y) = fx \land fy$ and $f1 = 1$.

Lemma (Hasimoto, 2001)

*If $[M]$ exists, then $[M]$ is a (dual normal) operator.*

\(^1\) Actually a dual normal operator
A unary map $f$ is an operator\(^1\) if $f(x \land y) = fx \land fy$ and $f1 = 1$.

**Lemma (Hasimoto, 2001)**

If $[M]$ exists, then $[M]$ is a (dual normal) operator.

**Lemma (Hasimoto, 2001)**

Assume that $M$ is finite, and every map in $M$ is an operator. Then $[M]$ exists, and

$$[M]x = \bigwedge \{fx \mid f \in M\}$$

---

\(^1\)Actually a dual normal operator
Constructing normal filter terms

A unary map $f$ is an operator\(^1\) if $f(x \land y) = fx \land fy$ and $f1 = 1$.

Lemma (Hasimoto, 2001)

*If $[M]$ exists, then $[M]$ is a (dual normal) operator.*

Lemma (Hasimoto, 2001)

*Assume that $M$ is finite, and every map in $M$ is an operator. Then $[M]$ exists, and*

$$[M]x = \bigwedge \{fx \mid f \in M\}$$

Lemma (T., 2016)

*If there exists a term $t$ in the language of $A$ such that $t^A x = [M]x$, then $t$ is a normal filter term.*

---

\(^1\)Actually a dual normal operator
Constructing normal filter terms

It is easy to show that if normal filter terms $t_1$ and $t_2$ exist for signatures $M_1$ and $M_2$ then $t_1 \land t_2$ is a normal filter term for $M_1 \cup M_2$, so we will redirect our focus towards normal filter terms for single functions.
Constructing normal filter terms

It is easy to show that if normal filter terms $t_1$ and $t_2$ exist for signatures $M_1$ and $M_2$ then $t_1 \land t_2$ is a normal filter term for $M_1 \cup M_2$, so we will redirect our focus towards normal filter terms for single functions.

Definition
Let $A$ be a Heyting algebra and let $f$ be a unary operation on $A$. The map $f$ is an anti-operator if $f(x \land y) = fx \lor fy$, and, $f1 = 0$. 
Constructing normal filter terms

It is easy to show that if normal filter terms \( t_1 \) and \( t_2 \) exist for signatures \( M_1 \) and \( M_2 \) then \( t_1 \land t_2 \) is a normal filter term for \( M_1 \cup M_2 \), so we will redirect our focus towards normal filter terms for single functions.

**Definition**

Let \( A \) be a Heyting algebra and let \( f \) be a unary operation on \( A \). The map \( f \) is an *anti-operator* if \( f(x \land y) = fx \lor fy \), and, \( f1 = 0 \).

**Lemma (T., 2016)**

Let \( A \) be an EHA and let \( f \) be an anti-operator on \( A \). Then \([f]\) exists, and

\[
[f]x = \neg fx,
\]

where \( \neg x = x \rightarrow 0 \).
Examples

Example (Meskhi, 1982)

**Heyting algebras with involution.** Let $A$ be a Heyting algebra equipped with a single unary operation $i$ that is a dual automorphism. The map $tx := \neg ix$ is a normal filter term on $A$. 
Example (Meskhi, 1982)

**Heyting algebras with involution.** Let $A$ be a Heyting algebra equipped with a single unary operation $i$ that is a dual automorphism. The map $tx := \neg ix$ is a normal filter term on $A$.

**Definition**

A unary operation $\sim$ on a lattice $A$ is a *dual pseudocomplement operation* if the following equivalence is satisfied for all $x \in A$:

$$x \lor y = 1 \iff y \geq \sim x.$$
Examples

Example (Meskhi, 1982)

**Heyting algebras with involution.** Let $A$ be a Heyting algebra equipped with a single unary operation $i$ that is a dual automorphism. The map $tx := \neg ix$ is a normal filter term on $A$.

Definition

A unary operation $\sim$ on a lattice $A$ is a *dual pseudocomplement operation* if the following equivalence is satisfied for all $x \in A$:

$$x \lor y = 1 \iff y \geq \sim x.$$

Example (Sankappanavar, 1985)

**Dually pseudocomplemented Heyting algebras.** Let $A$ be a Heyting algebra expanded by a dual pseudocomplement operation. Then $\neg \sim$ is a normal filter term on $A$. 
Double-Heyting algebras

Definition
A double-Heyting algebra is an EHA with \( M = \{ \div \} \), where \( \div \) is a binary operation satisfying

\[
x \lor y \geq z \iff y \geq z \div x.
\]
Double-Heyting algebras

**Definition**
A double-Heyting algebra is an EHA with $M = \{ \cdot^- \}$, where $\cdot^-$ is a binary operation satisfying

$$x \lor y \geq z \iff y \geq z \cdot^- x.$$ 

Observe that $1 \cdot^- x$ defines a dual pseudocomplement operation.

Theorem (Sankappanavar, 1985)
Congruences on a double-Heyting algebra are exactly the congruences of the $\langle \lor, \land, \rightarrow, \sim, 0, 1 \rangle$ term-reduct.

We can also prove directly that $[\cdot^-] = \neg \sim$, but it does not fall into the general cases seen earlier.
Double-Heyting algebras

Definition
A double-Heyting algebra is an EHA with $M = \{\prime\}$, where $\prime$ is a binary operation satisfying

$$x \lor y \geq z \iff y \geq z \prime x.$$  

Observe that $1 \prime x$ defines a dual pseudocomplement operation.

Theorem (Sankappanavar, 1985)
Congruences on a double-Heyting algebra are exactly the congruences of the $\langle \lor, \land, \rightarrow, \sim, 0, 1 \rangle$ term-reduct.
Double-Heyting algebras

Definition
A double-Heyting algebra is an EHA with $M = \{\cdot\}$, where $\cdot$ is a binary operation satisfying

$$x \lor y \geq z \iff y \geq z \cdot x.$$ 

Observe that $1 \cdot x$ defines a dual pseudocomplement operation.

Theorem (Sankappanavar, 1985)

Congruences on a double-Heyting algebra are exactly the congruences of the $\langle \lor, \land, \rightarrow, \sim, 0, 1 \rangle$ term-reduct.

We can also prove directly that $[\cdot] = \neg\sim$, but it does not fall into the general cases seen earlier.
Double-Heyting algebras

Definition
Let $A$ be an expanded double-Heyting algebra. For a filter $F \subseteq A$, let $\mathcal{I}(F) = \downarrow \sim F := \{ y \in A | (\exists x \in F) y \leq \sim x \}$. 

Theorem (T., 2016)
Let $A$ be a double-Heyting algebra and let $F$ be a normal filter of $A$.

$\Rightarrow$ The map $\mathcal{I}$ is an isomorphism from normal filters to the lattice of ideals closed under $\sim \neg$.

$\Rightarrow$ If $f$ is a unary order-preserving map then $F$ is closed under $f$ if and only if $\mathcal{I}(F)$ is closed under $\sim f \neg$.

$\Rightarrow$ $\mathcal{I}(F)$ is closed under $f$ if and only if $F$ is closed under $\neg f \sim$.

The above theorem holds for dually pseudocomplemented Heyting algebras as well.
Double-Heyting algebras

Definition
Let $A$ be an expanded double-Heyting algebra. For a filter $F \subseteq A$, let $\mathcal{I}(F) = \downarrow \sim F := \{ y \in A \mid (\exists x \in F) \, y \leq \sim x \}$. 

Theorem (T., 2016)
Let $A$ be a double-Heyting algebra and let $F$ be a normal filter of $A$. The map $\mathcal{I}$ is an isomorphism from normal filters to the lattice of ideals closed under $\sim \neg$. If $f$ is a unary order-preserving map then $F$ is closed under $f$ if and only if $\mathcal{I}(F)$ is closed under $\sim f \neg$, and, $\mathcal{I}(F)$ is closed under $f$ if and only if $F$ is closed under $\neg f \sim$. The above theorem holds for dually pseudocomplemented Heyting algebras as well.
Double-Heyting algebras

Definition
Let $A$ be an expanded double-Heyting algebra. For a filter $F \subseteq A$, let $\mathcal{I}(F) = \downarrow \sim F := \{ y \in A \mid (\exists x \in F) \ y \leq \sim x \}$.

Theorem (T., 2016)
Let $A$ be a double-Heyting algebra and let $F$ be a normal filter of $A$.

- The map $\mathcal{I}$ is an isomorphism from normal filters to the lattice of ideals closed under $\sim \neg$.
Double-Heyting algebras

Definition
Let $A$ be an expanded double-Heyting algebra. For a filter $F \subseteq A$, let $\mathcal{I}(F) = \downarrow \sim F := \{ y \in A \mid (\exists x \in F) y \leq \sim x \}$.

Theorem (T., 2016)
Let $A$ be a double-Heyting algebra and let $F$ be a normal filter of $A$.

- The map $\mathcal{I}$ is an isomorphism from normal filters to the lattice of ideals closed under $\sim \neg$.
- If $f$ is a unary order-preserving map then
  - $F$ is closed under $f$ if and only if $\mathcal{I}(F)$ is closed under $\sim f \neg$, and,
  - $\mathcal{I}(F)$ is closed under $f$ if and only if $F$ is closed under $\neg f \sim$.

The above theorem holds for dually pseudocomplemented Heyting algebras as well.
Double-Heyting algebras

**Definition**

Let $\mathbf{A}$ be an expanded double-Heyting algebra. For a filter $F \subseteq A$, let $\mathcal{I}(F) = \downarrow \sim F := \{ y \in A \mid (\exists x \in F) y \leq \sim x \}$.

**Theorem (T., 2016)**

Let $\mathbf{A}$ be a double-Heyting algebra and let $F$ be a normal filter of $\mathbf{A}$.

- The map $\mathcal{I}$ is an isomorphism from normal filters to the lattice of ideals closed under $\sim \neg$.

- If $f$ is a unary order-preserving map then
  - $F$ is closed under $f$ if and only if $\mathcal{I}(F)$ is closed under $\sim f \neg$, and,
  - $\mathcal{I}(F)$ is closed under $f$ if and only if $F$ is closed under $\neg f \sim$.

The above theorem holds for dually pseudocomplemented Heyting algebras as well.
Double-Heyting algebras

- $F$ is closed under $f$ if and only if $\mathcal{I}(F)$ is closed under $\sim f \neg$.
- $\mathcal{I}(F)$ is closed under $f$ if and only if $F$ is closed under $\neg f \sim$.

Open problem

The proof of the above very explicitly relies on the operation $·−$ in the signature. Does the result still apply if we dispose of it?
Double-Heyting algebras

- $F$ is closed under $f$ if and only if $\mathcal{I}(F)$ is closed under $\sim f \neg$.
- $\mathcal{I}(F)$ is closed under $f$ if and only if $F$ is closed under $\neg f \sim$.

Theorem (T., 2016)

Let $A$ be an EHA and assume $\cdot \neg \in M$. Let $f$ be a unary operation on $A$

- If $f$ preserves joins and $f0 = 0$ then $\neg f \sim x$ is a normal filter term for $f$.
- If $f$ reverses joins and $f1 = 0$ then $\neg \sim f \sim x$ is a normal filter term for $f$. 

Open problem

The proof of the above very explicitly relies on the operation $\cdot \neg$ in the signature. Does the result still apply if we dispose of it?
Double-Heyting algebras

- $F$ is closed under $f$ if and only if $\mathcal{I}(F)$ is closed under $\sim f \neg$.
- $\mathcal{I}(F)$ is closed under $f$ if and only if $F$ is closed under $\neg f \sim$.

**Theorem (T., 2016)**

*Let $A$ be an EHA and assume $\cdot \neg \in M$. Let $f$ be a unary operation on $A$*

- If $f$ preserves joins and $f0 = 0$ then $\neg f \sim x$ is a normal filter term for $f$.
- If $f$ reverses joins and $f1 = 0$ then $\neg \sim f \sim x$ is a normal filter term for $f$.

**Open problem**

The proof of the above very explicitly relies on the operation $\cdot \neg$ in the signature. Does the result still apply if we dispose of it?
Lemma

Let $\mathbf{A}$ be an EHA, let $t$ be a normal filter term on $\mathbf{A}$, and let $dx = x \wedge tx$. Then $(y, 1) \in Cg^\mathbf{A}(x, 1)$ if and only if $y \geq d^n x$ for some $n \in \omega$. 
Subdirectly irreducibles
Subdirectly irreducibles

\[ x \circ t x \]
Subdirectly irreducibles
Subdirectly irreducibles
Subdirectly irreducibles
Subdirectly irreducibles

Lemma
Let $A$ be an EHA, let $t$ be a normal filter term on $A$, and let $d^x = x \land tx$. Then $(y, 1) \in Cg^A(x, 1)$ if and only if $y \geq d^n x$ for some $n \in \omega$.

Lemma
Let $A$ be an EHA, let $t$ be a normal filter term on $A$, and let $d^x = x \land tx$.

1. $A$ is subdirectly irreducible if and only if there exists $b \in A \setminus \{1\}$ such that for all $x \in A \setminus \{1\}$ there exists $n \in \omega$ such that $d^n x \leq b$.

2. $A$ is simple if and only if for all $x \in A \setminus \{1\}$ there exists $n \in \omega$ such that $d^n x = 0$. 
Definition
A variety $\mathcal{V}$ has *definable principal congruences* (DPC) if there exists a first-order formula $\varphi(x, y, z, w)$ in the language of $\mathcal{V}$ such that, for all $A \in \mathcal{V}$, and all $a, b, c, d \in A$, we have

$$(a, b) \in Cg^A(c, d) \iff A \models \varphi(a, b, c, d).$$

If $\varphi$ is a finite conjunction of equations then $\mathcal{V}$ has *equationally definable principal congruences* (EDPC).
Definition
A variety $\mathcal{V}$ has *definable principal congruences* (DPC) if there exists a first-order formula $\varphi(x, y, z, w)$ in the language of $\mathcal{V}$ such that, for all $A \in \mathcal{V}$, and all $a, b, c, d \in A$, we have

$$ (a, b) \in Cg^A(c, d) \iff A \models \varphi(a, b, c, d). $$

If $\varphi$ is a finite conjunction of equations then $\mathcal{V}$ has *equationally definable principal congruences* (EDPC).

Theorem (T., 2016)
Let $\mathcal{V}$ be a variety of EHAs with a common normal filter term $t$, and let $dx = x \land tx$. The following are equivalent:

1. $\mathcal{V}$ has EDPC,
2. $\mathcal{V}$ has DPC,
3. $\mathcal{V} \models d^{n+1}x = d^nx$ for some $n \in \omega$. 
Discriminator varieties

Definition
A variety is *semisimple* if every subdirectly irreducible member of $\mathcal{V}$ is simple. If there is a ternary term $t$ in the language of $\mathcal{V}$ such that $t$ is a discriminator term on every subdirectly irreducible member of $\mathcal{V}$, i.e.,

$$t(x, y, z) = \begin{cases} x & \text{if } x \neq y \\ z & \text{if } x = y \end{cases},$$

then $\mathcal{V}$ is a *discriminator variety*. 

Theorem (Blok, Köhler and Pigozzi, 1984)
Let $\mathcal{V}$ be a variety of any signature. The following are equivalent:
1. $\mathcal{V}$ is semisimple, congruence permutable, and has EDPC.
2. $\mathcal{V}$ is a discriminator variety.
Discriminator varieties

Definition
A variety is *semisimple* if every subdirectly irreducible member of $\mathcal{V}$ is simple. If there is a ternary term $t$ in the language of $\mathcal{V}$ such that $t$ is a discriminator term on every subdirectly irreducible member of $\mathcal{V}$, i.e.,

$$t(x, y, z) = \begin{cases} x & \text{if } x \neq y \\ z & \text{if } x = y, \end{cases}$$

then $\mathcal{V}$ is a *discriminator variety*.

Theorem (Blok, Köhler and Pigozzi, 1984)
Let $\mathcal{V}$ be a variety of any signature. The following are equivalent:

1. $\mathcal{V}$ is semisimple, congruence permutable, and has EDPC.
2. $\mathcal{V}$ is a discriminator variety.
Discriminator varieties

Theorem (T., 2016)

Let $\mathcal{V}$ be a variety of dually pseudocomplemented EHA$s$, assume $\mathcal{V}$ has a normal filter term $t$, and let $dx = \neg\neg x \land tx$. The following are equivalent.

1. $\mathcal{V}$ is semisimple.
2. $\mathcal{V}$ is a discriminator variety.
3. $\mathcal{V}$ has DPC and $\exists m \in \omega$ such that $\mathcal{V} \models x \leq d\neg d^m x$.
4. $\mathcal{V}$ has EDPC and $\exists m \in \omega$ such that $\mathcal{V} \models x \leq d\neg d^m x$.
5. $\mathcal{V} \models d^{n+1} x = d^n x$ and $\mathcal{V} \models d\neg d^n x = \neg d^n x$ for some $n$

This generalises a result by Kowalski and Kracht (2006) for BAOs.