Expansions of Heyting algebras

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- **Double-Heyting algebras.** Double-Heyting algebras have their congruences determined by filters closed under the map

$$dx = (1 \dot{-} x) \to 0$$

Preliminaries

Definition

An algebra $\mathbf{A} = \langle A; M, \vee, \wedge, \rightarrow, 0, 1 \rangle$ is an *expanded Heyting algebra* (EHA) if the reduct $\langle A, \vee, \wedge, \rightarrow, 0, 1 \rangle$ is a Heyting algebra and M is a set of operations on A.

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- Let $x \leftrightarrow y = x \rightarrow y \wedge y \rightarrow x$. Recall that if $\mathbf{A}$ is a Heyting algebra and $F \subseteq A$ is a filter, then the binary relation $\theta(F) = \{(x, y) \mid x \leftrightarrow y \in F\}$ is a congruence on $\mathbf{A}$.

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A filter $F \subseteq A$ is *compatible* with an n-ary operation f on A if

$$\{x_i \leftrightarrow y_i \mid i \leq n\} \subseteq F \text{ implies } f(\vec{x}) \leftrightarrow f(\vec{y}) \in F.$$

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Theorem

If $\mathbf{A}$ is an EHA then $\theta(F)$ is a congruence on $\mathbf{A}$ if and only if F is compatible with f for every $f \in M$.

Normal filter terms

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Let t be a unary term in the language of **A**. We say that t is a *normal filter term* (*on* **A**) provided that it is order-preserving, and for every filter F of **A**, the filter F is a normal filter if and only if F is closed under $t^{\mathbf{A}}$.

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Example

The identity function is a normal filter term for unexpanded Heyting algebras.

Normal filter terms

Hence, the algebras from before have normal filter terms.

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$$dx = f_1 x \wedge f_2 x \wedge \ldots f_n x$$
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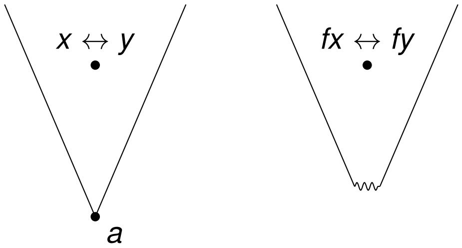
Let us say that a class of similar algebras has a normal filter term t if t is a normal filter term for each of those algebras.

Constructing normal filter terms

Let f be an n-ary operation on A. For each $a \in A$, define the set

$$f^{\leftrightarrow}(a) = \{f(\vec{x}) \leftrightarrow f(\vec{y}) \mid (\forall i \leq n)\ x_i, y_i \in A \text{ and } x_i \leftrightarrow y_i \geq a\}.$$



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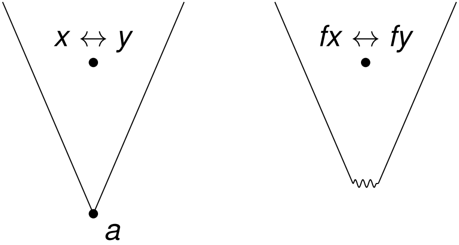

Now define the partial operation $[M]$ by

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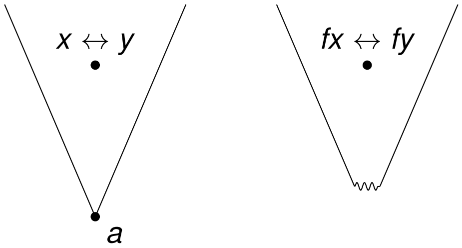

Now define the partial operation $[M]$ by

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If it is defined everywhere then we say that $[M]$ *exists in* **A**.

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A unary map f is an *operator*[1] if $f(x \wedge y) = fx \wedge fy$ and $f1 = 1$.

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If $[M]$ *exists, then* $[M]$ *is a* (*dual normal*) *operator.*

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Assume that M is finite, and every map in M is an operator. Then $[M]$ *exists, and*

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Lemma (T., 2016)

If there exists a term t in the language of $\mathbf{A}$ such that $t^{\mathbf{A}}x = [M]x$, then t is a normal filter term.

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Constructing normal filter terms

It is easy to show that if normal filter terms t_1 and t_2 exist for signatures M_1 and M_2 then $t_1 \wedge t_2$ is a normal filter term for $M_1 \cup M_2$, so we will redirect our focus towards normal filter terms for single functions.

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Definition

Let $\mathbf{A}$ be a Heyting algebra and let f be a unary operation on A. The map f is an *anti-operator* if $f(x \wedge y) = fx \vee fy$, and, $f1 = 0$.

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Lemma (T., 2016)

Let $\mathbf{A}$ be an EHA and let f be an anti-operator on A. Then $[f]$ exists, and

$$[f]x = \neg fx,$$

where $\neg x = x \to 0$.

Examples

Example (Meskhi, 1982)

Heyting algebras with involution. Let **A** be a Heyting algebra equipped with a single unary operation i that is a dual automorphism. The map $tx := \neg ix$ is a normal filter term on **A**.

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A unary operation $\sim$ on a lattice **A** is a *dual pseudocomplement operation* if the following equivalence is satisfied for all $x \in A$:

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Example (Sankappanavar, 1985)

Dually pseudocomplemented Heyting algebras. Let **A** be a Heyting algebra expanded by a dual pseudocomplement operation. Then $\neg{\sim}$ is a normal filter term on **A**.

Double-Heyting algebras

Definition

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Theorem (Sankappanavar, 1985)

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We can also prove directly that $[\dot{-}] = \neg\sim$, but it does not fall into the general cases seen earlier.

Double-Heyting algebras

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Let **A** be an expanded double-Heyting algebra. For a filter $F \subseteq A$, let $\mathcal{I}(F) = {\downarrow}{\sim}F := \{y \in A \mid (\exists x \in F)\ y \leq {\sim}x\}$.

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- *If f is a unary order-preserving map then*
  - *F is closed under f if and only if $\mathcal{I}(F)$ is closed under ${\sim}f\neg$, and,*
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The above theorem holds for dually pseudocomplemented Heyting algebras as well.

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Let $\mathbf{A}$ be an EHA and assume $\dot{-} \in M$. Let f be a unary operation on A

- *If f preserves joins and $f0 = 0$ then $\neg f \sim x$ is a normal filter term for f.*
- *If f reverses joins and $f1 = 0$ then $\neg \sim f \sim x$ is a normal filter term for f.*

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Open problem

The proof of the above very explicitly relies on the operation $\dot{-}$ in the signature. Does the result still apply if we dispose of it?

Subdirectly irreducibles

Lemma

Let $\mathbf{A}$ *be an EHA, let* t *be a normal filter term on* $\mathbf{A}$*, and let* $dx = x \wedge tx$*. Then* $(y, 1) \in \mathrm{Cg}^{\mathbf{A}}(x, 1)$ *if and only if* $y \geq d^n x$ *for some* $n \in \omega$*.*

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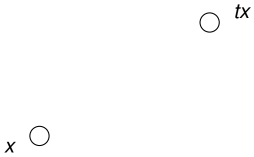

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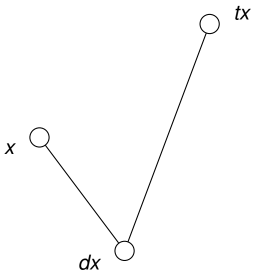

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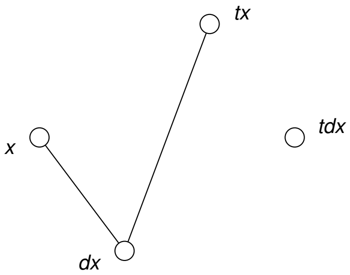

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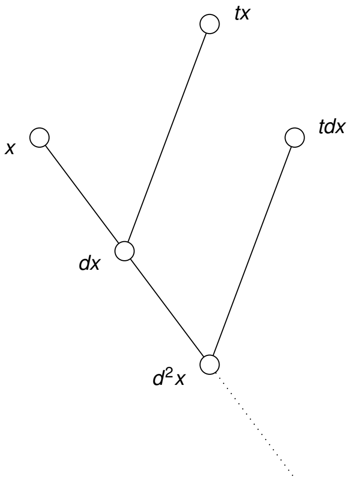

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Lemma

Let $\mathbf{A}$ *be an EHA, let* t *be a normal filter term on* $\mathbf{A}$*, and let* $dx = x \wedge tx$*.*

1. $\mathbf{A}$ *is subdirectly irreducible if and only if there exists* $b \in A \backslash \{1\}$ *such that for all* $x \in A \backslash \{1\}$ *there exists* $n \in \omega$ *such that* $d^n x \leq b$*.*
2. $\mathbf{A}$ *is simple if and only if for all* $x \in A \backslash \{1\}$ *there exists* $n \in \omega$ *such that* $d^n x = 0$*.*

EDPC

Definition

A variety $\mathcal{V}$ has *definable principal congruences* (DPC) if there exists a first-order formula $\varphi(x, y, z, w)$ in the language of $\mathcal{V}$ such that, for all $\mathbf{A} \in \mathcal{V}$, and all $a, b, c, d \in A$, we have

$$(a, b) \in \mathrm{Cg}^{\mathbf{A}}(c, d) \iff \mathbf{A} \models \varphi(a, b, c, d).$$

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Theorem (T., 2016)

Let $\mathcal{V}$ be a variety of EHAs with a common normal filter term t, and let $dx = x \wedge tx$. The following are equivalent:

1. *$\mathcal{V}$ has EDPC,*
2. *$\mathcal{V}$ has DPC,*
3. *$\mathcal{V} \models d^{n+1}x = d^n x$ for some $n \in \omega$.*

Discriminator varieties

Definition

A variety is *semisimple* if every subdirectly irreducible member of $\mathcal{V}$ is simple. If there is a ternary term t in the language of $\mathcal{V}$ such that t is a discriminator term on every subdirectly irreducible member of $\mathcal{V}$, i.e.,

$$t(x, y, z) = \begin{cases} x & \text{if } x \neq y \\ z & \text{if } x = y, \end{cases}$$

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Theorem (Blok, Köhler and Pigozzi, 1984)

Let $\mathcal{V}$ be a variety of any signature. The following are equivalent:

1. *$\mathcal{V}$ is semisimple, congruence permutable, and has EDPC.*
2. *$\mathcal{V}$ is a discriminator variety.*

Discriminator varieties

Theorem (T., 2016)

Let $\mathcal{V}$ be a variety of dually pseudocomplemented EHAs, assume $\mathcal{V}$ has a normal filter term t, and let $dx = \neg{\sim}x \wedge tx$. The following are equivalent.

1. *$\mathcal{V}$ is semisimple.*
2. *$\mathcal{V}$ is a discriminator variety.*
3. *$\mathcal{V}$ has DPC and $\exists m \in \omega$ such that $\mathcal{V} \models x \leq d{\sim}d^m\neg x$.*
4. *$\mathcal{V}$ has EDPC and $\exists m \in \omega$ such that $\mathcal{V} \models x \leq d{\sim}d^m\neg x$.*
5. *$\mathcal{V} \models d^{n+1}x = d^n x$ and $\mathcal{V} \models d{\sim}d^n x = {\sim}d^n x$ for some n*

This generalises a result by Kowalski and Kracht (2006) for BAOs.