Definability and Conceptual Completeness for Regular Logic

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The general answer is: When the theories, seen as categories, have equivalent completions of some kind (respectively, when we have some kind of quotient between completions of the theories)

In particular we focus on *regular* theories: They comprise sentences of the form $\forall\vec{x}(\varphi(\vec{x}) \rightarrow \psi(\vec{x}))$, where φ, ψ are built from atomic formulae by $\exists$, $\wedge$.

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Adding quotients of equivalence relations in a conservative way so that every equivalence relation is the kernel pair of its coequalizer.

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For any effective category $\mathcal{E}$, any regular functor $F\colon \mathcal{D} \to \mathcal{E}$,

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\zeta_{\mathcal{D}}"] \arrow[dr, "F"'] & \mathcal{D}_{ef} \arrow[d, "F^*"] \\ & \mathcal{E}, \end{tikzcd}$$

F^* regular, unique up to natural iso.

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Objects are quotients in the topos $yD_1 \underset{yd_1}{\overset{yd_0}{\rightrightarrows}} yD_0 \overset{e}{\twoheadrightarrow} X$ of equivalence relations coming from $\mathcal{D}$.

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Hongde Hu (corollary to a Stone - type duality for accessible categories): When $I_{ef}\colon \mathbb{T}_{ef} \to \mathbb{T}'_{ef}$ is covering, full on subobjects.

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M. Makkai, G. Reyes (and possibly J. Giraud): regular + conservative, full and covering between **effective categories** $\Rightarrow$ equivalence

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Proof: The (hard) "only if" part follows from the result of Hu. I_{ef} is covering and full on subobjects and these properties are reflected to I. The "if" part follows from the fact that if the properties hold for I, they are preserved by $(-)_{ef}$.

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The least obvious preservation result, that is of some independent interest is

Main Lemma: If $F\colon \mathcal{C} \to \mathcal{D}$ is a full on subobjects regular functor then $F^* = F_{ef} : \mathcal{C}_{ef} \to \mathcal{D}_{ef}$ is also full on subobjects.

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Proof: For a subobject $\sigma\colon S \to F^*X$ let the presentation

$$FC_1 \underset{Fc_1}{\overset{Fc_0}{\rightrightarrows}} FC_0 \xrightarrow{F^*e} F^*X$$

of F^*X, arise from one of X in $\mathcal{C}_{ef}$.

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Pull back the subobject S along F^*e to obtain by our assumption a subobject $Fi\colon FR_0 \to FC_0$, for a subobject $i\colon R_0 \to C_0$, and a regular epimorphism $s\colon FR_0 \to S$. Define the equivalence relation $(r_0, r_1)\colon R_1 \to R_0 \times R_0$ as the intersection of $(c_0, c_1)\colon C_1 \to C_0 \times C_0$ with the subobject $R_0 \times R_0 \to C_0 \times C_0$.

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Its coequalizer $\zeta_{\mathcal{C}} R_1 \underset{r_1}{\overset{r_0}{\rightrightarrows}} \zeta_{\mathcal{C}} R_0 \xrightarrow{q} \!\!\!\!\rightarrow\ Q$ in $\mathcal{C}_{ef}$ gives $S \cong F^*Q$.

$$\begin{tikzcd}
 & & F^*Q \\
FR_1 \arrow[r, shift left, "Fr_0"] \arrow[r, shift right, "Fr_1"'] \arrow[d, "Fj"] & FR_0 \arrow[r, two heads, "s"] \arrow[ur, two heads, "F^*q"'] \arrow[d, "Fi"] & S \arrow[d, "\sigma"] \\
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We find that $s \cdot Fr_0 = s \cdot Fr_1$, hence a regular epi $r\colon F^*Q \to S$ with $r \cdot F^*q = s$. Suffices that it is also a mono:

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F^*q is a regular epi so the "elements" u_0, u_1 are locally in $\zeta_{\mathcal{D}} D_i$: There is a covering $d' \colon D' \to D$, $i = 0, 1$ and factorizations $u_i \cdot d' = F^*q \cdot v_i$.

$$
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D' \arrow[r, two heads, "d'"] \arrow[dr, shift left, "v_0"] \arrow[dr, shift right, "v_1"'] & D \arrow[r, shift left, "u_0"] \arrow[r, shift right, "u_1"'] & F^*Q \arrow[d, "r"] \\
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Well-known examples: $\mathbf{CHaus} \simeq \mathbf{Stone}_{ef}$, $\mathbf{AbGr} \simeq \mathbf{TFAbGr}_{ef}$.

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Having a quotient between regular categories amounts to extending the theory of the domain by new axioms without adding new symbols:

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Corollary: If a regular theory $\mathbb{S}$ extends another one $\mathbb{T}$ by adding axioms, then $\mathbb{S}^{eq}$ extends $\mathbb{T}^{eq}$ by adding axioms.