

Definability and Conceptual Completeness for Regular Logic

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The general answer is: When the theories, seen as categories, have equivalent completions of some kind (respectively, when we have some kind of quotient between completions of the theories)

In particular we focus on *regular* theories: They comprise sentences of the form $\forall \vec{x}(\varphi(\vec{x}) \rightarrow \psi(\vec{x}))$, where φ, ψ are built from atomic formulae by \exists, \wedge .

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Adding quotients of equivalence relations in a conservative way so that every equivalence relation is the kernel pair of its coequalizer.

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Objects are quotients in the topos $yD_1 \xrightarrow[yd_1]{yd_0} yD_0 \xrightarrow{e} \gg X$ of equivalence relations coming from \mathcal{D} .

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Hongde Hu (corollary to a Stone - type duality for accessible categories):

When $I_{ef}: \mathbb{T}_{ef} \rightarrow \mathbb{T}'_{ef}$ is covering, full on subobjects.

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M. Makkai, G. Reyes (and possibly J. Giraud): regular + conservative, full and covering between **effective categories** \Rightarrow equivalence

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Theorem: A regular functor $I: \mathbb{T} \rightarrow \mathbb{T}'$ induces a fully faithful inclusion $-\cdot I: \text{Reg}(\mathbb{T}', \text{Set}) \rightarrow \text{Reg}(\mathbb{T}, \text{Set})$ iff I is covering, full on subobjects (for short: I is a quotient).

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The least obvious preservation result, that is of some independent interest is

Main Lemma: If $F: \mathcal{C} \rightarrow \mathcal{D}$ is a full on subobjects regular functor then $F^* = F_{ef} : \mathcal{C}_{ef} \rightarrow \mathcal{D}_{ef}$ is also full on subobjects.

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Proof: For a subobject $\sigma: S \rightarrow F^*X$ let the presentation

$$FC_1 \begin{array}{c} \xrightarrow{F_{C_0}} \\ \xrightarrow{F_{C_1}} \end{array} FC_0 \xrightarrow{F^*e} F^*X \text{ of } F^*X, \text{ arise from one of } X \text{ in } \mathcal{C}_{ef}.$$

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Pull back the subobject S along F^*e to obtain by our assumption a subobject $Fi: FR_0 \rightarrow FC_0$, for a subobject $i: R_0 \rightarrow C_0$, and a regular epimorphism $s: FR_0 \rightarrow S$. Define the equivalence relation $(r_0, r_1): R_1 \rightarrow R_0 \times R_0$ as the intersection of $(c_0, c_1): C_1 \rightarrow C_0 \times C_0$ with the subobject $R_0 \times R_0 \rightarrow C_0 \times C_0$.

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Its coequalizer $\zeta_{\mathcal{C}} R_1 \begin{array}{c} \xrightarrow{r_0} \\ \xrightarrow{r_1} \end{array} \zeta_{\mathcal{C}} R_0 \xrightarrow{q} Q$ in \mathcal{C}_{ef} gives $S \cong F^*Q$.

$$\begin{array}{ccccc}
 & & & & F^*Q \\
 & & & \nearrow F^*q & \\
 FR_1 & \xrightarrow{Fr_0} & FR_0 & \xrightarrow{s} & S \\
 & \xrightarrow{Fr_1} & & & \downarrow \sigma \\
 \downarrow Fj & & \downarrow Fi & & F^*X \\
 FC_1 & \xrightarrow{Fc_0} & FC_0 & \xrightarrow{F^*e} & \\
 & \xrightarrow{Fc_1} & & &
 \end{array}$$

$$\begin{array}{ccccc}
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We find that $s \cdot Fr_0 = s \cdot Fr_1$, hence a regular epi $r: F^*Q \rightarrow S$ with $r \cdot F^*q = s$. Suffices that it is also a mono:

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$$\begin{array}{ccccc}
 D' & \xrightarrow{d'} \twoheadrightarrow D & \xrightarrow{u_0} F^*Q & & \\
 & \searrow v_0 & \nearrow u_1 & \downarrow r & \\
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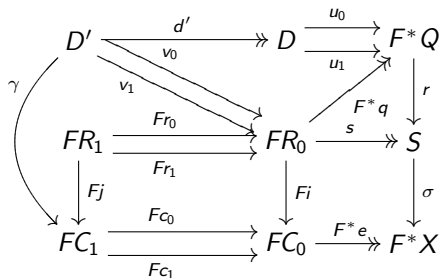
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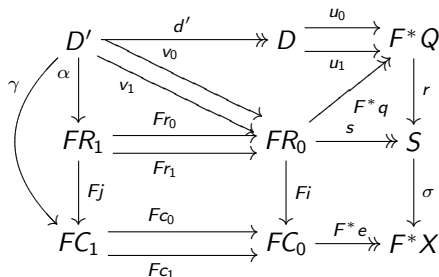
F^*q is a regular epi so the "elements" u_0, u_1 are locally in $\zeta_D D_i$: There is a covering $d': D' \rightarrow D$, $i = 0, 1$ and factorizations $u_i \cdot d' = F^*q \cdot v_i$.

$$\begin{array}{ccccc}
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 & & & &
 \end{array}$$

The diagram illustrates a complex commutative structure involving several objects and morphisms. The objects are arranged in a grid-like fashion, with morphisms connecting them in various directions. The morphisms are labeled with letters and subscripts, indicating specific relationships between the objects. The diagram is a representation of a mathematical proof or a theoretical result, likely related to the Main Result mentioned in the header.

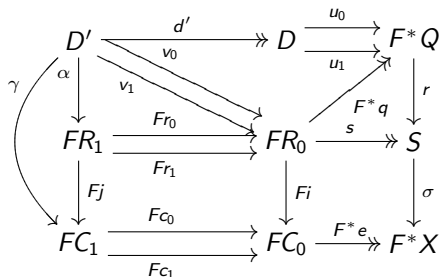


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Well-known examples: $\mathbf{CHaus} \simeq \mathbf{Stone}_{ef}$, $\mathbf{AbGr} \simeq \mathbf{TFAbGr}_{ef}$.

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Corollary: If a regular theory \mathbb{S} extends another one \mathbb{T} by adding axioms, then \mathbb{S}^{eq} extends \mathbb{T}^{eq} by adding axioms.