Twist products and dualities

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- Goal: Simplify twist product constructions by finding more transparent presentations for operations.
- Plan: Construct a topological duality appropriately tailored to a simplified rendering of (a particular variant of) the twist product construction.
- Then transport the twist product across this duality

Residuated lattices

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Examples:

CRLs generalize Boolean algebras, Heyting algebras, abelian lattice-ordered groups, MV-algebras, etc.

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Definition:

The expansion of a CRL by a unary operation $\neg$ satisfying $\neg\neg a = a$ and $a \to \neg b = b \to \neg a$ is called an *involutive* CRL. The expansion of a (involutive) CRL by constants designating universal bounds for its lattice reduct is called a *bounded* (involutive) CRL.

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Abelian lattice-ordered groups are involutive CRLs, where the involution is the group inverse. Boolean algebras are involutive CRLs, where the involution is the operation of complementation.

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Remark:

Throughout this talk, we work with bounded Sugihara monoids.

Residuated lattices (cont.)

Set $\mathbf{S} = \mathbb{Z} \cup \{-\infty, \infty\}$. If we impose the obvious order on $\mathbf{S}$ and define

$$a \cdot b = \begin{cases} a & |a| > |b| \\ b & |a| < |b| \\ a \wedge b & |a| = |b| \end{cases}$$

and

$$a \to b = \begin{cases} (-a) \vee b & a \leq b \\ (-a) \wedge b & a \not\leq b \end{cases}$$

Then $\mathbf{S}$ gives a Sugihara monoid with identity 0 and $\neg$ being $-$.

Residuated lattices (cont.)

$\mathbf{S} \setminus \{0\}$ is closed under the operations of **S** except for the identity. It turns out that 1 is an identity with respect to every element of **S** except for 0, so $\mathbf{S} \setminus \{0\}$ is a Sugihara monoid with the same operations as **S** except with identity 1.

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Proposition:

The Sugihara monoids are generated as a quasivariety by $\{\mathbf{S}, \mathbf{S} \setminus \{0\}\}$.

An important consequence of this is that every Sugihara monoid satisfies the identity $a \wedge \neg a \leq b \vee \neg b$.

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The negative elements of any Sugihara monoid form a Heyting algebra that satisfies the additional identity

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The Gödel algebra of negative elements of a Sugihara monoid is called its *negative cone*, and it turns out we can recover the entire Sugihara monoid from its negative cone and just a little bit more data via a variant of the twist product construction.

Definition:

We call a Gödel algebra with an additional constant f satisfying the identity

$$a \vee (a \to f) = t$$

a *Gödel algebra with Boolean constant* (or bG-algebra for short).

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Given a Sugihara monoid $\mathbf{A}$, let $\mathbf{A}_{\bowtie}$ be the negative cone of $\mathbf{A}$ with an additional designated constant given by $\neg t$. Then $\mathbf{A}_{\bowtie}$ is a bG-algebra.

Given a bG-algebra $\mathbf{A} = (A, \wedge, \vee, \to, \bot, \top, f)$, let

$$A^{\bowtie} = \{(a, b) \in A \times A : a \wedge b \leq f \text{ and } a \vee b = \top\}$$

We endow $A^{\bowtie}$ with an order $\sqsubseteq$ given by

$$(a, b) \sqsubseteq (c, d) \text{ iff } a \leq c \text{ and } b \geq d$$

and this order has corresponding lattice operations $\sqcap$ and $\sqcup$.

Recovering Sugihara monoids (cont.)

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This lattice has greatest element $(\top, \bot)$, least element $(\bot, \top)$, and an involution $\neg$ given by $\neg(a, b) = (b, a)$.

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We will define operations $\cdot$ and $\Rightarrow$ that make $A^{\bowtie}$ into a Sugihara monoid.

Define auxillary operations on A by $a \rightharpoonup b = (a \wedge f) \to b$ and $a^* = a \to f$.

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Define $\cdot$ by $(a_1, b_1) \cdot (a_2, b_2) = (a_3, b_3)$, where

$$a_3 = [(a_1 \rightharpoonup b_2) \wedge (a_2 \rightharpoonup b_1)] \to (a_1 \wedge b_2),$$

and

$$\begin{aligned} b_3 =& [(a_1 \rightharpoonup b_2) \wedge (a_2 \rightharpoonup b_1)] \\ & \wedge [((a_1 \rightharpoonup b_2) \wedge (a_2 \rightharpoonup b_1)) \to (a_1 \wedge a_2)]^* \end{aligned}$$

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Define $\Rightarrow$ by $(a_1, b_1) \Rightarrow (a_2, b_2) = (a_3, b_3)$, where

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$$\begin{aligned} b_3 =& [((a_1 \to a_2) \wedge (b_2 \rightharpoonup b_1)) \rightharpoonup (a_1 \wedge (t \rightharpoonup b_2)] \\ & \wedge [(a_1 \to a_2) \wedge (b_2 \rightharpoonup b_1)]^* \end{aligned}$$

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Starting with a bG-algebra $\mathbf{A}$, we obtain that $\mathbf{A}^{\bowtie}$ is a Sugihara monoid with the aforedefined operations.

Recovering Sugihara monoids (cont.)

If SM and bG are the categories of Sugihara monoids and bG-algebras, respectively, then $(-)_{\bowtie} \colon \mathsf{SM} \to \mathsf{bG}$ and $(-)^{\bowtie} \colon \mathsf{bG} \to \mathsf{SM}$ may be lifted to functors.

Theorem:

$(-)_{\bowtie}$ and $(-)^{\bowtie}$ witness a covariant equivalence of categories between SM and bG.

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Theorems of the above improve our understanding of how ordered algebras can be constructed from simple, more familiar ones. However, generalizing results like this is difficult due to the complexity of the construction. Is there some way to recast the construction in more intuitive terms?

Urquhart duality

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Definition:

An *relevant space* is a structure of the form $(\mathbf{X}, R, {}', I)$, where $\mathbf{X} = (X, \leq, \tau)$ is a Priestley space, R is a ternary relation on X, $'$ is a unary function on X, and $I \subseteq X$, all satsifying some conditions stated presently. To state these conditions, we for $U, V \subseteq X$ define

$$U \bullet V = \{z : (\exists x, y)(Rxyz \text{ and } x \in U \text{ and } y \in V)\}$$

$$U \rightsquigarrow V = \{x : (\forall y, z)((Rxyz \text{ and } y \in U) \text{ implies } z \in V)\}$$

Urquhart duality (cont.)

Definition (cont.):

The axioms of a relevant space are

1. If U, V are clopen up-sets, then $U \bullet V$ and $U \to V$ are clopen,
2. If $Rxyz$, $u \leq x$, $v \leq y$, and $z \leq w$, then $Ruvw$,
3. For all $x, y, z \in X$, if $Rxyz$, then there exist clopen up-sets U, V such that $x \in U$, $y \in V$, and $z \notin U \bullet V$,
4. $'$ is continuous and order-reversing.

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4. $'$ is continuous and order-reversing.

Additional properties for classes of algebras may be translated along the Urquhart duality in a simple way. Given x, y in a relevant space $(\mathbf{X}, R, {}', I)$, set

$$x \odot y = \{z \in X : Rxyz\}$$

Urquhart duality (cont.)

Definition:

A relevant space $(\mathbf{X}, R, ', I)$ is called a *Sugihara relevant space* if it satisfies

1. $x'' = x$,
2. $x \odot y = y \odot x$,
3. $x \odot (y \odot z) = (x \odot y) \odot z$,
4. $x \odot x = x$, and
5. If $z \in x \odot y$, then $y' \in x \odot z'$.

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As their name suggests, Sugihara relevant spaces are precisely the relevant spaces corresponding to Sugihara monoids.

The duality between Sugihara relevant spaces and Sugihara monoids functions as follows. Given a Sugihara monoid $\mathbf{A}$, let $\mathbf{A}_*$ be the Priestley space of prime filters of $\mathbf{A}$ considered as a distributive lattice. For $x, y \in \mathbf{A}_*$, let $x \cdot y = {\uparrow}\{ab : a \in x, b \in y\}$.

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Define a ternary relation R on $\mathbf{A}_*$ by $Rxyz$ iff $x \cdot y \subseteq z$, define a unary operation $'$ on $\mathbf{A}_*$ by $x' = \{a \in A : \neg a \notin x\}$, and set $I = \{x \in \mathbf{A}_* : t \in x\}$. These definitions give a Sugihara relevant space $(\mathbf{A}_*, R, ', I)$.

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On the other hand, for a Sugihara relevant space $(\mathbf{X}, R, {}', I)$, let $\mathbf{X}^*$ be dual of $\mathbf{X}$ as a Priestley space (i.e., its bounded distributive lattice of clopen up-sets).

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Then the operations $\bullet$, $\to$, I, and $\neg U = \{x : x' \notin U\}$ turn $\mathbf{X}^*$ into a Sugihara monoid.

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Theorem:

The category of Sugihara monoids is dually equivalent to the category of Sugihara-Esakia spaces, with morphisms the Esakia maps that preserve D and $D^{\complement}$.

The Esakia duality for Sugihara monoids is based on the Davey-Werner natural duality for Kleene algebras instead of the Priestley duality. Define an algebra **K** with universe $\{-1, 0, 1\}$, lattice operations coming from the order $-1 < 0 < 1$, and the involution $\neg a = -a$.

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Define also another order on K by $-1 \sqsubseteq 0$ and $1 \sqsubseteq 0$, and set $D(\underset{\sim}{\mathbf{K}}) = \{-1, 1\}$

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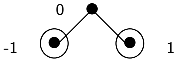

Denote by $\underset{\sim}{\mathbf{K}}$ the ordered topological space with underlying set $\{-1, 0, 1\}$, order $\sqsubseteq$, the discrete topology, and designated subset $D(\underset{\sim}{\mathbf{K}})$.

Esakia duality for Sugihara monoids (cont.)

For a Sugihara monoid $\mathbf{A}$, let $\mathbf{A}_+$ be the collection of $(\wedge, \vee, \neg, \bot, \top)$-homomorphisms into $\mathbf{K}$. Give $\mathbf{A}_+$ the order, topology, and designated subset $D(\mathbf{A}_+)$ inherited pointwise from $\underset{\sim}{\mathbf{K}}$. This makes $\mathbf{A}_+$ into Sugihara-Esakia space.

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For a Sugihara-Esakia space $\mathbf{X}$, let $\mathbf{X}^+$ be the collection of continuous monotone maps preserving $D(\mathbf{X})$ from $\mathbf{X}$ into $\underset{\sim}{\mathbf{K}}$. Give $\mathbf{X}^+$ the lattice operations and involution inherited pointwise from $\mathbf{K}$. With some additional operations defined by terms as before, this makes $\mathbf{X}^+$ into a Sugihara monoid.

Dualized twist products

Our strategy to simplify the construction $(-)^{\bowtie}$: Transport the construction along the Urquhart duality and Esakia duality for Sugihara monoids.

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More specifically, let $\mathbf{X}$ be a Sugihara-Esakia space. We construct a new structured topological space that we call (by abuse of notation) $\mathbf{X}^{\bowtie}$ as follows.

Make a new copy of those elements of X outside of $D(\mathbf{X})$:

$$-D(\mathbf{X}) = \{-x : x \notin D(\mathbf{X})\}$$

Set $X^{\bowtie} = X \cup -D(\mathbf{X})$. Endow $\mathbf{X}^{\bowtie}$ with the disjoint union topology τ coming from X and $-D(\mathbf{X})$ (considered as a subspace of $\mathbf{X}$).

Dualized twist products (cont.)

We put additional structure on $X^{\bowtie}$ as follows. First, put an order $\leq^{\bowtie}$ on $X^{\bowtie}$ by the conditions

- If $x, y \in X$, then $x \leq^{\bowtie} y$ if and only if $x \leq y$,
- If $-x, -y \in -X$, then $-x \leq^{\bowtie} -y$ if and only if $y \leq x$,
- If $-x \in -X$ and $y \in X$, then $-x \leq^{\bowtie} y$ if and only if x is comparable to y in X.

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Extend $-$ to an operation on $X^{\bowtie}$ by stipulating that $-(-x) = x$ for $-x \in -D(\mathbf{X})$.

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Set $I(X^{\bowtie}) = X$.

Dualized twist products (cont.)

Finally, define the notion of the absolute value of an element of $X^{\bowtie}$ in the obvious way, and define a partial binary operation $\cdot$ on $X^{\bowtie}$ by

$$x \cdot y = \begin{cases} x \vee y & \text{if } x \parallel y, \text{ provided the join exists} \\ y & \text{if } x \perp y \text{ and } |x| < |y| \\ x & \text{if } x \perp y \text{ and } |y| < |x| \\ x \wedge y & \text{if } x \perp y \text{ and } |x| = |y| \\ \text{undefined} & \text{otherwise} \end{cases}$$

where $\perp$ denotes the relation of comparability and $\parallel$ denotes the relation of incomparability. For $x, y, z \in X^{\bowtie}$, define $Rxyz$ iff $x \cdot y$ exists and $x \cdot y \leq^{\bowtie} z$. The resulting structure $\mathbf{X}^{\bowtie}$ is a Sugihara relevant space.

Dualized negative cones

On the other hand, if $\mathbf{X} = (X, \leq, \tau, R, {}', I)$ is a Sugihara relevant space, set $\mathbf{X}_{\bowtie} = (I, \leq, D, \tau_I)$, where

- $\leq$ is the order of $\mathbf{X}$ restricted to I,
- τ_I is the topology on I as a subspace of $\mathbf{X}$, and
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Moreover, $(-)_{\bowtie}$ and $(-)^{\bowtie}$ can be lifted to functors between the category of Sugihara-Esakia spaces and the category of Sugihara relevant spaces.

Lifting to morphisms

Let SES be the category of Sugihara-Esakia spaces with morphisms the continuous Esakia maps preserving both D and $D^{\complement}$. Let SRS be the category of Sugihara relevant spaces with morphisms the *relevant maps* between them, i.e., maps $\varphi\colon \mathbf{X} \to \mathbf{Y}$ between Sugihara relevant spaces with

1. φ is continuous and isotone,
2. If $R_{\mathbf{X}}xyx$, then $R_{\mathbf{Y}}\varphi(x)\varphi(y)\varphi(z)$,
3. If $R_{\mathbf{Y}}xy\varphi(z)$, then there exists $u, v \in X$ such that $R_{\mathbf{X}}uvz$, $x \leq \varphi(u)$, and $y \leq \varphi(v)$.
4. If $R_{\mathbf{Y}}\varphi(x)yz$, then there exists $u, v \in X$ such that $R_{\mathbf{X}}xuv$, $y \leq \varphi(u)$, and $\varphi(v) \leq z$,
5. $\varphi(x') = \varphi(x)'$, and
6. $\varphi^{-1}[I_{\mathbf{Y}}] = I_{\mathbf{X}}$.

Lifting to morphisms (cont.)

If $\varphi\colon \mathbf{X} \to \mathbf{Y}$ is a morphism of SRS, let $\varphi_{\bowtie}\colon \mathbf{X}_{\bowtie} \to \mathbf{Y}_{\bowtie}$ by $\varphi_{\bowtie} = \varphi \restriction_{X_{\bowtie}}$. Then $\varphi_{\bowtie}$ is a morphism of SES.

Lifting to morphisms (cont.)

If $\varphi\colon \mathbf{X} \to \mathbf{Y}$ is a morphism of SRS, let $\varphi_{\bowtie}\colon \mathbf{X}_{\bowtie} \to \mathbf{Y}_{\bowtie}$ by $\varphi_{\bowtie} = \varphi \restriction_{X_{\bowtie}}$. Then $\varphi_{\bowtie}$ is a morphism of SES.

On the other hand, for $\varphi\colon \mathbf{X} \to \mathbf{Y}$ a morphism of SES, define $\varphi^{\bowtie}\colon \mathbf{X}^{\bowtie} \to \mathbf{Y}^{\bowtie}$ by

$$\varphi^{\bowtie}(x) = \begin{cases} \varphi(x) & \text{if } x \in X, \\ -\varphi(-x) & \text{if } x \in -D^{\complement} \end{cases}$$

Then $\varphi^{\bowtie}$ is a morphism of SRS.

The main theorem

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This provides a vast simplification of the algebraic functors described previously.

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- First, for a Sugihara monoid $\mathbf{A}$, show that $\mathbf{A}_*$ and $(\mathbf{A}_+)^{\bowtie}$ are isomorphic Priestley spaces,
- Second, characterize the ternary relation R on $\mathbf{A}_*$ and show that it coincides with the one given by our partial binary operation on $(\mathbf{A}_+)^{\bowtie}$.

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- Second, characterize the ternary relation R on $\mathbf{A}_*$ and show that it coincides with the one given by our partial binary operation on $(\mathbf{A}_+)^{\bowtie}$.

We consider only some key pieces. First, $\mathbf{A}_+$ and $\{x \in A_* : t \in x\}$ are order-isomorphic via the map $h \mapsto h^{-1}[\{0,1\}]$, and this map can be extended to an isomorphism of Priestley spaces due to the following fact.

Lemma:

Let $\mathbf{A}$ be a Sugihara monoid and let $x \in A_*$. Then $t \in x$ or $t \in x'$, and $t \in x, x'$ iff $x = x'$.

Proof of the main theorm (cont.)

The harder part of the proof consists of characterizing the ternary relation. The following fact is crucial.

Lemma:

Let **A** be a Sugihara monoid and let $x, y \in A_*$. Then $x \cdot y = \uparrow\{ab : a \in x, b \in y\}$ is either a prime filter of A_* or else is A itself.

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This lemma allows us to work with the (partial) binary operation $\cdot$ on $\mathbf{A}_*$ instead of with the ternary relation. We consider an example to illustrate the flavor of the proofs.

Proof of the main theorem (cont.)

Lemma:

Let $\mathbf{A}$ be a Sugihara monoid and let $x \in A_*$. Then $x' \cdot x = x \wedge x'$.

Proof:

Either $x \subseteq x'$ or $x' \subseteq x$, so without loss of generality assume $x' \subseteq x$. Then $e \in x$, so $x' \subseteq x' \cdot x$. On the other hand, let $c \in x' \cdot x$. Then there exists $a \in x'$ and $b \in x$ with $ab \leq c$. This holds iff $b \cdot \neg c \leq \neg a$. If $\neg c \in x$, then $b \cdot \neg c \leq \neg a$ would give $\neg a \in x$, a contradiction to $a \in x'$. Hence $\neg c \notin x$, so $c \in x'$. Thus $x' \cdot x \subseteq x'$ and $x' \cdot x = x \wedge x'$.

Consequences

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It also has a number of both algebraic and logical consequences. For example, it resolves the question of how the Dunn and Routley-Meyer relational semantics for the logic **R**-mingle are connected (the answer: via the topological version of the twist product).

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It also has a number of both algebraic and logical consequences. For example, it resolves the question of how the Dunn and Routley-Meyer relational semantics for the logic **R**-mingle are connected (the answer: via the topological version of the twist product).

Similar techniques can be used to obtain dualized presentations of other complicated algebraic constructions.

Thank you!

For additional information, visit my website at
http://www.cs.du.edu/˜wesfussn/.