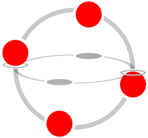
A Duality for Boolean Contact Algebras

Julien Raskin



TACL 2017

Boolean contact algebras

A (extensional) Boolean contact algebra (BCA) is a Boolean algebra B endowed with a binary relation $\mathcal{C}$ s.t.

- C0 $a \perp 0$
- C1 $a \neq 0 \Rightarrow a \,\mathcal{C}\, a$ (reflexivity)
- C2 $a \,\mathcal{C}\, b \Rightarrow b \,\mathcal{C}\, a$ (symmetry)
- C3 $a \,\mathcal{C}\, b \leq c \Rightarrow a \,\mathcal{C}\, c$
- C4 $a \,\mathcal{C}\, (b \vee c) \Rightarrow a \,\mathcal{C}\, b$ or $a \,\mathcal{C}\, c$
- C5 $a \not\leq b \Rightarrow \exists c\; a \,\mathcal{C}\, c$ and $c \perp b$ (extensionality)

Boolean contact algebras

Example

If (X, τ) is a semiregular topological space, then $\mathrm{RC}(X)$ is a complete *non-extensional* BCA.

- $F \vee G = F \cup G$
- $F \wedge G = (F \cap G)^{\circ -}$
- $\neg F = F^{c-}$
- $F \mathrel{\mathcal{C}} G \Leftrightarrow F \cap G \neq \emptyset$

It satisfies C5 iff (X, τ) is weakly regular.

Theorem (Düntsch-Winter, 2005)

Every BCA can be densely embedded into the BCA $\mathrm{RC}(X)$ for some T_1 weakly regular topological space X.

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Theorem (Düntsch-Winter, 2005)

Every BCA can be densely embedded into the BCA $\mathrm{RC}(X)$ for some T_1 weakly regular topological space X.

The Representation Theorem

A subset Γ of B is a *clan* if

- $a \in \Gamma, a \leq b \Rightarrow b \in \Gamma$
- $a \vee b \in \Gamma \Rightarrow a \in \Gamma$ or $b \in \Gamma$
- $1 \in \Gamma$
- $a, b \in \Gamma \Rightarrow a \, \mathcal{C} \, b$.

A maximal clan is called a *cluster*.

- Every clan is contained in a cluster.
- If $a \, \mathcal{C} \, b$, there exists a clan Γ s.t. $a, b \in \Gamma$.
- Every ultrafilter is a clan.
- Every clan is a union of ultrafilters.

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We define on the set $\mathrm{clu}(B)$ of clusters of B the topology having the sets

$$\Box b = \{\Gamma \in \mathrm{clu}(B) : \neg b \notin \Gamma\}$$

as a basis.

Then $\mathrm{clu}(B)$ is T_1 and weakly regular, and

$$\eta_B : B \to \mathrm{RC}(\mathrm{clu}(B)) : b \mapsto \Diamond b = \{\Gamma \in \mathrm{clu}(B) : b \in \Gamma\}$$

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de Vries duality

A *de Vries algebra* is a complete Boolean algebra endowed with a binary relation $\prec$ satisfying

DV0 $0 \prec a \prec 1$

DV1 $a \prec b \Rightarrow a \leq b$ (reflexivity)

DV2 $a \prec b \Rightarrow \neg b \prec \neg a$ (symmetry)

DV3 $a \leq b \prec c \leq d \Rightarrow a \prec d$

DV4 $a \prec b, c \Rightarrow a \prec b \wedge c$

DV5 $b \neq 0 \Rightarrow \exists a \neq 0 \; a \prec b$ (extensionality)

DV6 $a \prec b \Rightarrow \exists c \; a \prec c \prec b$ (transitivity)

If $a \; \mathcal{C} \; b \Leftrightarrow a \not\prec \neg b$, then DV0-DV5 correspond to C0-C5. The axiom DV6 correspond to

C6 $a \perp b \Rightarrow \exists c \; a \perp c$ and $\neg c \perp b$

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A filter $\mathcal{F}$ is *round* if $b \in \mathcal{F} \Rightarrow \exists a \in \mathcal{F}\ a \prec b$. An *end* is a maximal round filter. The set $\mathrm{end}(B)$ of ends is endowed with the topology having the sets

$$r_B(b) = \{\mathcal{F} \in \mathrm{end}\, b : b \in \mathcal{F}\}$$

as a basis. This space is compact Hausdorff.

If X is a compact Hausdorff space, then the set $\mathrm{RO}(X)$ of regular open sets of X, endowed with the relation $\prec$ defined by $U \prec V \Rightarrow \bar{U} \subseteq V$, is a de Vries algebra.

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A map $\alpha : B \to B'$ is a *de Vries morphism* if it satisfies

DVM1 $\alpha(0) = 0$

DVM2 $\alpha(a \wedge b) = \alpha(a) \wedge \alpha(b)$

DVM3 $a \prec b \Rightarrow \neg\alpha(\neg a) \prec \alpha(b)$

DVM4 $\alpha(b) = \bigvee\{\alpha(a) : a \prec b\}$.

If α is a de Vries morphism, then the map

$$f_\alpha : \mathrm{end}(B') \to \mathrm{end}(B) : \mathcal{F}' \mapsto \alpha^{-1}(\mathcal{F}') \Uparrow$$

is continuous.

If $f : X' \to X$ is a continuous map, then

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The composition of two de Vries morphisms is defined by

$$(\alpha_2 \star \alpha_1)(b) = (\alpha_2 \circ \alpha_1)^\star(b) = \bigvee \{\alpha_2(\alpha_1(a)) : a \prec b\}.$$

Then, the category **DeV** of de Vries algebras is dually equivalent to the category **KHaus** of compact Hausdorff spaces.

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The map

$$\theta_B : \mathrm{clu}(B) \to \mathrm{end}(B) : \Gamma \mapsto \{b \in B : \neg b \notin \Gamma\}$$

is a homeomorphism.

A subordination $\prec$ on B yields a closed relation on the Stone dual of B.

If $(B, \prec)$ is a de Vries algebra, then R is an equivalence relation. There is a 1-1 correspondence between the equivalence classes and the clusters.

Bezhanishvili et al. established (2016) a duality for Boolean algebras with subordinations.

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A little digression

Modal-like duality

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Every BCA can be densely embedded into $\mathrm{RC}(X)$ for some T_1 weakly regular topological space X.

Corollary

Every complete BCA is isomorphic to $\mathrm{RC}(X)$ for some T_1 weakly regular topological space X.

Our aim is to turn this representation theorem into a duality.

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If B is complete, then the clans of $\mathrm{clu}(B)$ are fixed: if γ is a clan of $\mathrm{RC}(\mathrm{clu}(B))$, then $\bigcap \gamma \neq \emptyset$.

A topological space Y is a *cluster space* if it is T_1, weakly regular and if its clans are fixed.

If Y is a cluster space, the map

$$\varepsilon_Y : Y \to \mathrm{clu}(\mathrm{RC}(Y)) : y \mapsto \{F \in \mathrm{RC}(Y) : y \in F\}$$

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A map $\beta : B \to B'$ between two BCAs is a *contact morphism* if

CM1 $\beta(1) = 1$

CM2 $\beta(a \vee b) = \beta(a) \vee \beta(b)$

CM3 $a \perp b \Rightarrow \beta(a) \perp \beta(b)$.

The inverse image of a clan under a contact morphism is a clan.

Dual: $N_\beta : \mathrm{clu}(B') \to \mathrm{clu}(B)$?

Towards a duality

Morphisms

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N_\beta \quad &: \quad \mathrm{clu}(B') \to \mathrm{clan}(\mathrm{RC}(\mathrm{clu}(B))) \\
&: \quad \Gamma' \mapsto \{F \in \mathrm{RC}(\mathrm{clu}(B)) : \beta(\eta_B^{-1}(F)) \in \Gamma'\}
\end{aligned}
$$

We define on $\mathrm{clan}(\mathrm{RC}(Y))$ the topology which has the family $\{\{\gamma \in \mathrm{clan}(RC(Y)) : F \in \gamma\} : F \in \mathrm{RC}(Y)\}$ as a basis for closed sets.

The inverse image of a regular closed set under N_β is regular closed.

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If Y, Y' are cluster spaces, a morphism from Y' to Y is a map

$$N : Y' \to \mathrm{clan}(\mathrm{RC}(Y))$$

s.t. the inverse image of a regular closed set is regular closed.
Then

$$\beta_N : \mathrm{RC}\, Y \to \mathrm{RC}\, Y' : F \mapsto N^{-1}(\{\gamma : F \in \gamma\})$$

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The composition of $N : Y' \to \mathrm{clan}(\mathrm{RC}(Y))$ and $N' : Y'' \to \mathrm{clan}(\mathrm{RC}(Y'))$ is defined by

$$(N \star N')(y'') = \beta_N^{-1}(N'(y'')).$$

- **EBCA**: category of complete extensional BCAs with contact morphisms (and normal composition)
- **CluSp**: category of cluster spaces with cluster spaces morphisms and composition $\star$

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Generalizing de Vries duality

$\alpha = \neg\beta(\neg\,\cdot)$ de Vries morphism between B and B':

$$\begin{aligned} N_\beta &: \mathrm{clu}(B') \to \mathrm{clan}(\mathrm{RC}(\mathrm{clu}(B))) \\ f_\alpha &: \mathrm{clu}(B') \to \mathrm{clu}(B) \end{aligned}$$

Then,

$$N_\beta(\Gamma') \subseteq \varepsilon_{\mathrm{clu}(B)}(f_\alpha(\Gamma')).$$

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Generalizing de Vries duality

- Not the same composition!
- $\beta : B \to B'$ satisfying CM1-4
- $\beta' \star \beta$ defined as in de Vries duality
- Corresponding cluster space morphisms with corresponding composition
- Dual equivalence between the two modified categories extending de Vries duality

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Duality through clans

From now, we will consider complete BCAs satisfying C0-C4.

Let's equip $\mathrm{clan}(B)$ with the topology having $\{\Box b : b \in B\}$ as a basis, where $\Box b = \{\Gamma \in \mathrm{clan}(B) : \neg b \notin \Gamma\}$.

Theorem

The space $\mathrm{clan}(B)$ is semiregular, sober, and its clans are full.

The clans of Z are full if

$$\forall \gamma \in \mathrm{clan}(\mathrm{RC}(Z))\ \forall F \in \mathrm{RC}(Z)\ \left(\bigcap \gamma \subseteq F \Rightarrow F \in \gamma\right).$$

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Duality through clans

A map between two clan spaces is a clan space morphism if the inverse image of a regular closed set is regular closed.
If $\beta : B \to B'$ is a contact morphism, then

$$h_\beta : \mathrm{clan}(B') \to \mathrm{clan}(B) : \Gamma' \mapsto \beta^{-1}(\Gamma')$$

is a clan space morphism.
If $h : Y' \to Y$ is a clan space morphism, then

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$$\beta_h : \mathrm{RC}(Y) \to \mathrm{RC}(Y') : F \mapsto h^{-1}(F)$$

is a contact morphism.

Duality through clans

If B is a BCA, the map

$$\eta_B : B \to \mathrm{RC}(\mathrm{clan}(B)) : b \mapsto \{\Gamma \in \mathrm{clan}(B) : b \in \Gamma\}$$

is a isomorphism.

If Z is a clan space, the map

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- **BCA**: category of complete BCAs with contact morphisms
- **ClanSp**: category of clan spaces with clan space morphisms

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Thank you for your attention!