A Duality for Boolean Contact Algebras

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Boolean contact algebras

A (extensional) Boolean contact algebra (BCA) is a Boolean algebra $B$ endowed with a binary relation $C$ s.t.

C0 $a \perp 0$

C1 $a \neq 0 \Rightarrow a C a$ (reflexivity)

C2 $a C b \Rightarrow b C a$ (symmetry)

C3 $a C b \leq c \Rightarrow a C c$

C4 $a C (b \lor c) \Rightarrow a C b$ or $a C c$

C5 $a \not\leq b \Rightarrow \exists c \; a C c$ and $c \perp b$ (extensionality)
Example

If \((X, \tau)\) is a semiregular topological space, then \(RC(X)\) is a complete non-extensional BCA.

- \(F \lor G = F \cup G\)
- \(F \land G = (F \cap G)^\circ\)
- \(\neg F = F^c\)
- \(F \subseteq G \iff F \cap G \neq \emptyset\)

It satisfies C5 iff \((X, \tau)\) is weakly regular.

Theorem (Düntsch-Winter, 2005)

Every BCA can be densely embedded into the BCA \(RC(X)\) for some \(T_1\) weakly regular topological space \(X\).
Boolean contact algebras

Example
If \((X, \tau)\) is a semiregular topological space, then \(RC(X)\) is a complete non-extensional BCA.

\[
\begin{align*}
\bullet & \quad F \lor G = F \cup G \\
\bullet & \quad F \land G = (F \cap G)^{\circ-} \\
\bullet & \quad \neg F = F^{c-} \\
\bullet & \quad F \sqsubset G \iff F \cap G \neq \emptyset
\end{align*}
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It satisfies C5 iff \((X, \tau)\) is weakly regular.

Theorem (Düntsch-Winter, 2005)
Every BCA can be densely embedded into the BCA \(RC(X)\) for some \(T_1\) weakly regular topological space \(X\).
The Representation Theorem

A subset $\Gamma$ of $B$ is a clan if
- $a \in \Gamma$, $a \leq b \Rightarrow b \in \Gamma$
- $a \lor b \in \Gamma \Rightarrow a \in \Gamma$ or $b \in \Gamma$
- $1 \in \Gamma$
- $a, b \in \Gamma \Rightarrow a \not\leq b$.

A maximal clan is called a cluster.

- Every clan is contained in a cluster.
- If $a \not\leq b$, there exists a clan $\Gamma$ s.t. $a, b \in \Gamma$.
- Every ultrafilter is a clan.
- Every clan is a union of ultrafilters.
The Representation Theorem

A subset $\Gamma$ of $\mathcal{B}$ is a **clan** if

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A subset $\Gamma$ of $B$ is a *clan* if

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The Representation Theorem

We define on the set \( \text{clu}(B) \) of clusters of \( B \) the topology having the sets

\[
\square b = \{ \Gamma \in \text{clu}(B) : \neg b \notin \Gamma \}
\]

as a basis.

Then \( \text{clu}(B) \) is \( T_1 \) and weakly regular, and

\[
\eta_B : B \to \text{RC}(\text{clu}(B)) : b \mapsto \diamond b = \{ \Gamma \in \text{clu}(B) : b \in \Gamma \}
\]

is a dense embedding.
The Representation Theorem

We define on the set \( \text{clu}(B) \) of clusters of \( B \) the topology having the sets

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\Box b = \{ \Gamma \in \text{clu}(B) : \neg b \notin \Gamma \}
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is a dense embedding.
A de Vries algebra is a complete Boolean algebra endowed with a binary relation \( \prec \) satisfying

\[
\begin{align*}
\text{DV0} & \quad 0 \prec a \prec 1 \\
\text{DV1} & \quad a \prec b \implies a \leq b \quad \text{(reflexivity)} \\
\text{DV2} & \quad a \prec b \implies \neg b \prec \neg a \quad \text{(symmetry)} \\
\text{DV3} & \quad a \leq b \prec c \leq d \implies a \prec d \\
\text{DV4} & \quad a \prec b, c \implies a \prec b \land c \\
\text{DV5} & \quad b \neq 0 \implies \exists a \neq 0 \quad a \prec b \quad \text{(extensionality)} \\
\text{DV6} & \quad a \prec b \implies \exists c a \prec c \prec b \quad \text{(transitivity)}
\end{align*}
\]

If \( a \not C b \iff a \not \not \prec b \), then DV0-DV5 correspond to C0-C5. The axiom DV6 correspond to

\[
\begin{align*}
\text{C6} & \quad a \perp b \implies \exists c \quad a \perp c \quad \text{and} \quad \neg c \perp b
\end{align*}
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de Vries duality

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\text{DV4} & \quad a \prec b, c \Rightarrow a \prec b \land c \\
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If $a \not\sim b \iff a \not\prec \neg b$, then DV0-DV5 correspond to C0-C5. The axiom DV6 correspond to

\begin{align*}
\text{C6} & \quad a \perp b \Rightarrow \exists c \ a \perp c \text{ and } \neg c \perp b
\end{align*}
A filter $\mathcal{F}$ is \textit{round} if $b \in \mathcal{F} \Rightarrow \exists a \in \mathcal{F} \ a \prec b$. An \textit{end} is a maximal round filter. The set $\text{end}(B)$ of ends is endowed with the topology having the sets

$$r_B(b) = \{ \mathcal{F} \in \text{end} b : b \in \mathcal{F} \}$$

as a basis. This space is compact Hausdorff.

If $X$ is a compact Hausdorff space, then the set $RO(X)$ of regular open sets of $X$, endowed with the relation $\prec$ defined by $U \prec V \Rightarrow \overline{U} \subseteq V$, is a de Vries algebra.
A filter $\mathcal{F}$ is *round* if $b \in \mathcal{F} \Rightarrow \exists a \in \mathcal{F} \ a \prec b$. An *end* is a maximal round filter. The set $\text{end}(B)$ of ends is endowed with the topology having the sets

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A map $\alpha : B \to B'$ is a *de Vries morphism* if it satisfies

- **DVM1** $\alpha(0) = 0$
- **DVM2** $\alpha(a \wedge b) = \alpha(a) \wedge \alpha(b)$
- **DVM3** $a \prec b \Rightarrow \neg \alpha(\neg a) \prec \alpha(b)$
- **DVM4** $\alpha(b) = \bigvee \{ \alpha(a) : a \prec b \}$.

If $\alpha$ is a de Vries morphism, then the map

$$f_\alpha : \text{end}(B') \to \text{end}(B) : \mathcal{F}' \mapsto \alpha^{-1}(\mathcal{F}') \uparrow$$

is continuous.

If $f : X' \to X$ is a continuous map, then

$$\alpha_f : \text{RO}(X) \to \text{RO}(X') : U \mapsto (f^{-1}(U))^{-\circ}$$

is a de Vries morphism.
de Vries duality

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The composition of two de Vries morphisms is defined by

$$(\alpha_2 \star \alpha_1)(b) = (\alpha_2 \circ \alpha_1)^*(b) = \bigvee \{\alpha_2(\alpha_1(a)) : a \preceq b\}.$$ 

Then, the category $\text{Dev}$ of de Vries algebras is dually equivalent to the category $\text{KHaus}$ of compact Hausdorff spaces.
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Then, the category $\textbf{DeV}$ of de Vries algebras is dually equivalent to the category $\textbf{KHaus}$ of compact Hausdorff spaces.
The map

$$\theta_B : \text{clu}(B) \to \text{end}(B) : \Gamma \mapsto \{ b \in B : \neg b \notin \Gamma \}$$

is a homeomorphism.
A subordination $\prec$ on $B$ yields a closed relation on the Stone dual of $B$.

If $(B, \prec)$ is a de Vries algebra, then $R$ is an equivalence relation. There is a 1-1 correspondence between the equivalence classes and the clusters. Bezhanishvili et al. established (2016) a duality for Boolean algebras with subordinations.
A little digression
Modal-like duality

A subordination $\prec$ on $B$ yields a closed relation on the Stone dual of $B$.
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Towards a duality

Goal

Theorem (Düntsch-Winter, 2005)
Every BCA can be densely embedded into $\mathbb{R}C(X)$ for some $T_1$ weakly regular topological space $X$.

Corollary
Every complete BCA is isomorphic to $\mathbb{R}C(X)$ for some $T_1$ weakly regular topological space $X$.

Our aim is to turn this representation theorem into a duality.
Towards a duality

**Goal**

**Theorem (Düntsch-Winter, 2005)**
Every BCA can be densely embedded into $RC(X)$ for some $T_1$ weakly regular topological space $X$.

**Corollary**
Every complete BCA is isomorphic to $RC(X)$ for some $T_1$ weakly regular topological space $X$.

Our aim is to turn this representation theorem into a duality.
Towards a duality
Characterizing the dual spaces

If $B$ is complete, then the clans of $\text{clu}(B)$ are fixed: if $\gamma$ is a clan of $\text{RC}(\text{clu}(B))$, then $\bigcap \gamma \neq \emptyset$.

A topological space $Y$ is a *cluster space* if it is $T_1$, weakly regular and if its clans are fixed.

If $Y$ is a cluster space, the map

$$\varepsilon_Y : Y \to \text{clu}(\text{RC}(Y)) : y \mapsto \{ F \in \text{RC}(Y) : y \in F \}$$

is a homeomorphism.
Towards a duality
Characterizing the dual spaces

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Towards a duality
Characterizing the dual spaces

If $B$ is complete, then the clans of $\text{clus}(B)$ are fixed: if $\gamma$ is a clan of $\text{RC}(\text{clus}(B))$, then $\bigcap \gamma \neq \emptyset$.

A topological space $Y$ is a cluster space if it is $T_1$, weakly regular and if its clans are fixed. If $Y$ is a cluster space, the map

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is a homeomorphism.
A map $\beta : B \to B'$ between two BCAs is a contact morphism if

**CM1** $\beta(1) = 1$

**CM2** $\beta(a \lor b) = \beta(a) \lor \beta(b)$

**CM3** $a \perp b \Rightarrow \beta(a) \perp \beta(b)$. 

The inverse image of a clan under a contact morphism is a clan.

Dual: $N_{\beta} : \text{clu}(B') \to \text{clu}(B)$?
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Morphisms

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Towards a duality

Morphisms

\[ N_\beta : \text{clu}(B') \to \text{clan}(\text{RC}(\text{clu}(B))) \]
\[ \Gamma' \mapsto \{ F \in \text{RC}(\text{clu}(B)) : \beta(\eta_B^{-1}(F)) \in \Gamma' \} \]

We define on \( \text{clan}(\text{RC}(Y)) \) the topology which has the family \( \{ \{ \gamma \in \text{clan}(\text{RC}(Y)) : F \in \gamma \} : F \in \text{RC}(Y) \} \) as a basis for closed sets.

The inverse image of a regular closed set under \( N_\beta \) is regular closed.
Towards a duality
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Towards a duality

Morphisms

If $Y, Y'$ are cluster spaces, a morphism from $Y'$ to $Y$ is a map

$$N : Y' \rightarrow \text{clan}(\text{RC}(Y))$$

s.t. the inverse image of a regular closed set is regular closed. Then

$$\beta_N : \text{RC } Y \rightarrow \text{RC } Y' : F \mapsto N^{-1}(\{\gamma : F \in \gamma\})$$

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is a contact morphism.
The composition of $N : Y' \to \text{clan}(\text{RC}(Y))$ and $N' : Y'' \to \text{clan}(\text{RC}(Y'))$ is defined by

$$(N \ast N')(y'') = \beta^{-1}_N(N'(y'')).$$
Towards a duality

The duality

- **EBCA**: category of complete extensional BCAs with contact morphisms (and normal composition)
- **CluSp**: category of cluster spaces with cluster spaces morphisms and composition

The categories **EBCA** and **CluSp** are dually equivalent.
Towards a duality

The duality

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The duality

- **EBCA**: category of complete extensional BCAs with contact morphisms (and normal composition)
- **CluSp**: category of cluster spaces with cluster spaces morphisms and composition

The categories **EBCA** and **CluSp** are dually equivalent.
\[ \alpha = \neg \beta(\neg \cdot) \] de Vries morphism between \( B \) and \( B' \):

\[
\begin{align*}
N_\beta &: \text{clu}(B') \to \text{clan}(\text{RC}(\text{clu}(B))) \\
f_\alpha &: \text{clu}(B') \to \text{clu}(B)
\end{align*}
\]

Then,

\[
N_\beta(\Gamma') \subseteq \varepsilon_{\text{clu}(B)}(f_\alpha(\Gamma')).
\]
Generalizing de Vries duality

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Generalizing de Vries duality

- Not the same composition!
  - \( \beta : B \to B' \) satisfying CM1-4
  - \( \beta' \star \beta \) defined as in de Vries duality
  - Corresponding cluster space morphisms with corresponding composition
  - Dual equivalence between the two modified categories extending de Vries duality
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- Dual equivalence between the two modified categories extending de Vries duality
From now, we will consider complete BCAs satisfying C0-C4. Let’s equip $\text{clan}(B)$ with the topology having $\{\Box b : b \in B\}$ as a basis, where $\Box b = \{\Gamma \in \text{clan}(B) : \neg b \notin \Gamma\}$.

**Theorem**
The space $\text{clan}(B)$ is semiregular, sober, and its clans are full. The clans of $Z$ are full if

$$\forall \gamma \in \text{clan}(\text{RC}(Z)) \forall F \in \text{RC}(Z) \left( \bigcap \gamma \subseteq F \Rightarrow F \in \gamma \right).$$
Duality through clans

From now, we will consider complete BCAs satisfying C0-C4. Let’s equip \( \text{clan}(B) \) with the topology having \( \{ \square b : b \in B \} \) as a basis, where \( \square b = \{ \Gamma \in \text{clan}(B) : \neg b \notin \Gamma \} \).

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A topological space is called a *clan space* if it is semiregular, sober and if its clans are full.

If $Z$ is a clan space, then $RC(Z)$ is a BCA.
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Duality through clans

A map between two clan spaces is a clan space morphism if the inverse image of a regular closed set is regular closed. If \( \beta : B \to B' \) is a contact morphism, then

\[
\begin{align*}
h_\beta : \text{clan}(B') & \to \text{clan}(B) : \Gamma' \mapsto \beta^{-1}(\Gamma')
\end{align*}
\]

is a clan space morphism. If \( h : Y' \to Y \) is a clan space morphism, then

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If $B$ is a BCA, the map

$$\eta_B : B \to \text{RC}(\text{clan}(B)) : b \mapsto \{\Gamma \in \text{clan}(B) : b \in \Gamma\}$$

is a isomorphism.

If $Z$ is a clan space, the map

$$\varepsilon_Z : Z \to \text{clan}(\text{RC}(Z)) : z \mapsto \{F \in \text{RC}(Z) : z \in F\}$$

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Duality through clans

- **BCA**: category of complete BCAs with contact morphisms
- **ClanSp**: category of clan spaces with clan space morphisms

The categories **BCA** and **ClanSp** are dually equivalent.
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Thank you for your attention!