Topologizing filters on rings of fractions RS^{-1} and congruence relations on FilR

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Outline

Definition: Let $\langle L; \leq, \cdot, e_L \rangle$ and $\langle L'; \leq, \cdot, e_{L'} \rangle$ be lattice ordered monoids. Let $\varphi : L \to L'$ be a mapping that satisfies the following conditions:

- $\varphi(a \vee b) = \varphi(a) \vee \varphi(b)$ for all $a, b \in L$;
- $\varphi(\bigvee X) = \bigvee_{x \in X} \varphi(x)$ for all nonempty $X \subseteq L$;
- $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$.

We call a map φ satisfying the above a homomorphism of lattice ordered monoids.

Denote by $\equiv_\varphi$ the congruence on L induced by φ. Thus

$$a \equiv_\varphi b \Leftrightarrow \varphi(a) = \varphi(b) \text{ for all } a, b \in L.$$

Denote by $[a]_{\equiv_\varphi}$ the equivalence class of a. That is

$$[a]_{\equiv_\varphi} = \{b \in L : \varphi(a) = \varphi(b)\}.$$

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Denote by $L/\equiv_\varphi$ the collection of all equivalence classes with respect to $\equiv_\varphi$, that is

$$L/\equiv_\varphi = \{[a]_{\equiv_\varphi} : a \in L\}.$$

- The equivalence class $[a]_{\equiv_\varphi}$ contains a largest element, namely $\bigvee[a]_{\equiv_\varphi}$.
- If $\{\equiv_\delta : \delta \in \Delta\}$ is a family of congruences on L, then $\bigcap \equiv_\delta$ is a congruence on L and we have a canonical embedding of lattice ordered monoids given by:

$$L/\bigcap_{\delta\in\Delta} \equiv_\delta \hookrightarrow \prod_{\delta\in\Delta}(L/\equiv_\delta)$$
$$[a]_{\bigcap_{\delta\in\Delta}\equiv_\delta} \mapsto \{[a]_{\equiv_\delta}\}_{\delta\in\Delta}.$$

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Change of rings

Through out this section unless otherwise mentioned:

- R and T are arbitrary rings.
- I is two sided ideals denoted by $I \trianglelefteq R$.

Proposition 2.1. Let $\varphi : R \to T$ a ring homo. Then the map $\varphi^* : \mathrm{Fil}\ T_T \to \mathrm{Fil}\ R_R$ given by

$$\varphi^*(\mathfrak{F}) = \{K \leq R_R : K \supseteq \varphi^{-1}[L] \text{ for some } L \in \mathfrak{F}\}$$

is a complete lattice homomorphism.

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Proposition 2.2. Let $I \trianglelefteq R$ and $\pi : R \overset{can.ring\ epi.}{\longrightarrow} R/I$ the canonical ring epimorphism. Then

$$\pi^*(\mathfrak{F} : \mathfrak{G}) = [\pi^*(\mathfrak{F}) : \pi^*(\mathfrak{G})] \cap \eta(I)$$

for all $\mathfrak{F}, \mathfrak{G} \in \mathrm{Fil}\,(R/I)_{R/I}$.

We can see that in this situation, for each $\mathcal{F} \in \mathrm{Fil}\,(R/I)_{R/I}$,

$$\begin{aligned} \pi^*(\mathcal{F}) &= \{K \leq R_R : K \supseteq \pi^{-1}[L] \text{ for some } L \in \mathcal{F}\} \\ &= \{K \leq R_R : K \supseteq I \text{ and } K/I \in \mathcal{F}\}. \end{aligned} \tag{1}$$

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In light of the previous definition and Proposition 2, we see that

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It is easily checked that $\pi_*(\mathfrak{F}) \in \mathrm{Fil}\,(R/I)_{R/I}, \forall \mathfrak{F} \in [0, \eta(I)]$.

Theorem 2.3.(Correspondence Theorem)

Let $I \trianglelefteq R$ and $\pi : R \to R/I$ the canonical ring epimorphism. Then π^* and π_* are mutually inverse complete lattice and monoid isomorphisms between $\mathrm{Fil}\,(R/I)_{R/I}$ and $\langle [0, \eta(I)]; :_I \rangle$. Hence, $[\mathrm{Fil}\,(R/I)_{R/I}]^{\mathrm{du}}$ and $\langle [0, \eta(I)]; :_I \rangle^{\mathrm{du}}$ are isomorphic complete lattice ordered monoids.

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Theorem 2.4. (Preservation Theorem) Let $I \trianglelefteq R$.

(a) If the monoid operation : on $\mathrm{Fil}\, R_R$ is commutative, then so is the corresponding monoid operation on $\mathrm{Fil}\,(R/I)_{R/I}$.

(b) If every $\mathfrak{F} \in \mathrm{Fil}\, R_R$ is idempotent, that is to say, $\mathfrak{F} : \mathfrak{F} = \mathfrak{F}$, then the same is true of every member of $\mathrm{Fil}\,(R/I)_{R/I}$.

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Topologizing filters on the ring of fractions RS^{-1}

- Throughout this section S is a multiplicative subset of a commutative ring R.

The mapping from $\operatorname{Id} R$ to $\operatorname{Id} RS^{-1}$ given by $I \mapsto IS^{-1}$ induces in turn a map from $\operatorname{Fil} R$ to $\operatorname{Fil} RS^{-1}$.

For each $\mathfrak{F} \in \operatorname{Fil} R$ define

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(b) The mapping $\hat{\varphi}_S : \mathrm{Fil}\, R \to \mathrm{Fil}\, RS^{-1}$ is onto.

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$$\mathfrak{F}_S \stackrel{\text{def}}{=} \{A \leq R : A \cap S \neq \emptyset\}. \tag{4}$$

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Theorem 3.5.
The map $\hat{\varphi}_S : [\mathrm{Fil}\, R]^{\mathrm{du}} \to [\mathrm{Fil}\, RS^{-1}]^{\mathrm{du}}$ is an onto homomorphism of lattice ordered monoids.

The map $\hat{\varphi}_S$ of Theorem 3.5 gives rise to a canonical congruence relation $\equiv_{\hat{\varphi}_S}$ on $\mathrm{Fil}\, R$ defined by:

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$$S_P \stackrel{\text{def}}{=} R \backslash P.$$

Lemma 3.6. If R is any commutative ring, then

$\bigcap\{\mathfrak{F}_{S_P} : P \in \operatorname{Spec_m} R\} = \{R\}$, the identity of $\operatorname{Fil} R$ w.r.t. the monoid operation.

Proposition 3.7. Let R be a commutative ring for which $\operatorname{Fil} R$ is commutative. Then $\bigcap\{\equiv_{\hat{\varphi}_{S_P}} : P \in \operatorname{Spec_m} R\}$ is the identity congruence on $\operatorname{Fil} R$, that is, for all $\mathfrak{F}, \mathfrak{G} \in \operatorname{Fil} R$,

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With reference to the above sequence of mappings, recall that by Theorem 3.5, the mapping

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The aforementioned subdirect decomposition thus takes $\mathfrak{F}$ in $\operatorname{Fil} R$ onto $\{\hat{\varphi}_{S_P}(\mathfrak{F})\}_{P \in \operatorname{Spec_m} R}$ in $\prod_{P \in \operatorname{Spec_m} R} \operatorname{Fil} R_P$.

Corollary 3.8 Let R be an arbitrary commutative ring. For all jansian $\mathfrak{F}, \mathfrak{G} \in \operatorname{Fil} R$, we have

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The above result tells us that the homomorphism $\hat{\varphi}_S : [\mathrm{Fil}\, R]^{\mathrm{du}} \to [\mathrm{Fil}\, RS^{-1}]^{\mathrm{du}}$ restricts to a homomorphism from $[\mathrm{Jans}\, R]^{\mathrm{du}}$ onto $[\mathrm{Jans}\, RS^{-1}]^{\mathrm{du}}$.

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(b) $\mathrm{Fil}\,R$ is commutative.

Cont...

Proposition 3.11. Let R be a Prüfer domain for which $\mathrm{Fil}\, R$ is commutative. Then R_P is a (noetherian) rank 1 discrete valuation domain for every maximal ideal P of R.

Proposition 3.12. Let R be a Prüfer domain for which $\mathrm{Fil}\, R$ is commutative. Then every nonzero prime ideal of R is maximal.

Theorem 3.13. The following statements are equivalent for a Prüfer domain R:

(a) R is noetherian and thus a Dedekind domain;

(b) $\mathrm{Fil}\, R$ is commutative.

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