Sahlqvist via translation

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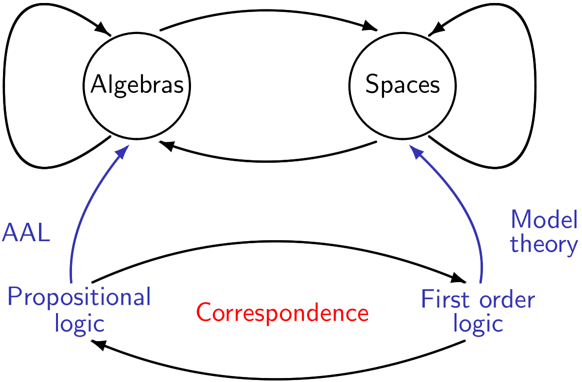
Generalised Sahlqvist theory

From a model theoretic problem to an algebraic logic problem



Unified correspondence

- DLE-logics [CP12, CPS]
- substructural logics [CP]
- hybrid logics [CR15]
- many valued modal logics [CIRM]
- mu calculus [CFPS15, CGP14, CC15]
- regular modal logics [PSZ16]
- possibility semantics [YZ]
- Jónsson-style vs Sambin-style canonicity [PSZ15]
- constructive canonicity [CP]
- Sahlqvist via translation [CPZ]
- constructive canonicity for lattice-based fixed point logics [CCPZ]
- display calculi [GMPTZ16]
- sequent calculi [MZ16]
- finite lattices and monotone modal logic [FPS16]

Correspondence and canonicity via translation

Question

Could correspondence and canonicity for non-classical (modal) logics be reduced to classical modal logic via e.g. Gödel-Tarski translations?

Answer

- Correspondence: yes, at least up to distributive lattice based logics.
- Canonicity: much more restricted.

An overview of Gödel-Tarski translation

Definition

$$\mathcal{L}_I \ni \varphi ::= p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi.$$

$$\mathcal{L}_{S4\Box} \ni \alpha ::= p \mid \bot \mid \top \mid \alpha \vee \alpha \mid \alpha \wedge \alpha \mid \neg\alpha \mid \Box_{\leq}\alpha.$$

The Gödel-Tarski translation is the map $\tau\colon \mathcal{L}_I \to \mathcal{L}_{S4\Box}$ defined by the following recursion:

$$\begin{array}{rcl}
\tau(p) & = & \Box_{\leq} p \\
\tau(\bot) & = & \bot \\
\tau(\top) & = & \top \\
\tau(\varphi \wedge \psi) & = & \tau(\varphi) \wedge \tau(\psi) \\
\tau(\varphi \vee \psi) & = & \tau(\varphi) \vee \tau(\psi) \\
\tau(\varphi \to \psi) & = & \Box_{\leq}(\neg\tau(\varphi) \vee \tau(\psi)).
\end{array}$$

An overview of Gödel-Tarski translation

Translation Lemma

For every intuitionistic formula φ and every partial order $\mathbb{F} = (W, \leq)$,

- $[\![\varphi]\!]_V = (\![\tau(\varphi)]\!)_V$ for every persistent valuation V on $\mathbb{F}$;
- $(\![\tau(\varphi)]\!)_U = [\![\varphi]\!]_{U^\uparrow}$ for every valuation U on $\mathbb{F}$;
- $\mathbb{F} \Vdash \varphi$ iff $\mathbb{F} \Vdash^* \tau(\varphi)$.

Example

In intuitionistic modal logic, take the formula $\Box\Diamond p \to \Diamond p$, its Gödel-Tarski translation is $\Box_{\leq}(\Box\Diamond\Box_{\leq} p \to \Diamond\Box_{\leq} p)$, which is not of the Sahlqvist shape.

One solution (c.f. Gehrke-Nagahashi-Venema 2005): take $\tau(p) = \Diamond_{\geq} p$, which translates $\Box\Diamond p \to \Diamond p$ into $\Box_{\leq}(\Box\Diamond\Diamond_{\geq} p \to \Diamond\Diamond_{\geq} p)$.

An algebraic perspective of view

- $\mathcal{L}_1$, $\mathcal{L}_2$: propositional languages over X
- $\mathbb{A}, \mathbb{B}$: ordered $\mathcal{L}_1, \mathcal{L}_2$- algebras
- $e \colon \mathbb{A} \hookrightarrow \mathbb{B}$:order-embedding
- $V \in \mathbb{A}^X$, $U \in \mathbb{B}^X$: valuations for proposition variables
- $[\![\cdot]\!]_V$ and $(\![\cdot]\!)_U$: valuations for $\mathcal{L}_1$, $\mathcal{L}_2$ formulas
- $\overline{e} : \mathbb{A}^X \to \mathbb{B}^X$: $V \mapsto e \circ V$

Algebraic translation lemma

Let $\tau \colon \mathcal{L}_1 \to \mathcal{L}_2$ and $r \colon \mathbb{B}^X \to \mathbb{A}^X$ be such that the following conditions hold for every $\varphi \in \mathcal{L}_1$:

(a) $e([\![\varphi]\!]_V) = (\![\tau(\varphi)]\!)_{\overline{e}(V)}$ for every $V \in \mathbb{A}^X$;

(b) $(\![\tau(\varphi)]\!)_U = e([\![\varphi]\!]_{r(U)})$ for every $U \in \mathbb{B}^X$.

Then, for all $\varphi, \psi \in \mathcal{L}_1$,

$$\mathbb{A} \models \varphi \leq \psi \quad \text{iff} \quad \mathbb{B} \models \tau(\varphi) \leq \tau(\psi).$$

Example 1: Gödel-Tarski translation

- $\mathcal{L}_1$=intuitionistic propositional language
- $\mathcal{L}_2$=propositional S4 modal language with $\Box_\leq$
- $\mathbb{A}$: Heyting algebra
- $\mathbb{B}$: S4 modal algebra constructed from $\mathbb{A}$
- $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ order-embedding (to be proved)
- $\iota\colon \mathbb{B} \to \mathbb{A}$ the right adjoint of $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ (existence to be proved)
- $\Box_\leq : \mathbb{B} \to \mathbb{B}$ defined as $e \circ \iota$ ($\mathbb{B}$ being a normal S4 modal algebra to be proved)
- translation τ defined as before
- $r : \mathbb{B}^X \to \mathbb{A}^X$: $U \mapsto \iota \circ U$
- conditions (a) and (b) hold for r and τ (to be proved, by induction)

Example 1: Gödel-Tarski translation

Existence of Boolean algebra and right adjoint

For every Heyting algebra $\mathbb{A}$, there exists a Boolean algebra $\mathbb{B}$ such that $\mathbb{A}$ embeds into $\mathbb{B}$ via some order-embedding $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ which is also a homomorphism of the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$ and has a right adjoint $\iota\colon \mathbb{B} \to \mathbb{A}$ verifying condition $a \to^{\mathbb{A}} b = \iota(\neg^{\mathbb{B}} e(a) \vee^{\mathbb{B}} e(b))$. Finally, these facts lift to the canonical extensions of $\mathbb{A}$ and $\mathbb{B}$ as in the following diagram:

$$
\begin{tikzcd}
\mathbb{A}^\delta \arrow[rr, hook, "e^\delta"'] \arrow[rr, phantom, bend left=20, "\top"] & & \mathbb{B}^\delta \arrow[ll, bend right=40, "\iota^\pi"'] \\
\\
\mathbb{A} \arrow[uu, hook] \arrow[rr, hook, "e"'] \arrow[rr, phantom, bend left=20, "\top"] & & \mathbb{B} \arrow[uu, hook] \arrow[ll, bend right=40, "\iota"']
\end{tikzcd}
$$

Example 1: Gödel-Tarski translation

Proof

Via Esakia duality, the Heyting algebra $\mathbb{A}$ can be identified with the algebra of clopen upsets of its associated Esakia space $\mathbb{X}_{\mathbb{A}}$, which is a Priestley space, hence a Stone space.
Let $\mathbb{B}$ be the Boolean algebra of the clopen subsets of $\mathbb{X}_{\mathbb{A}}$. Since any clopen up-set is in particular a clopen subset, a natural order embedding $e : \mathbb{A} \hookrightarrow \mathbb{B}$ exists, which is also a lattice homomorphism between $\mathbb{A}$ and $\mathbb{B}$. This shows the first part of the claim.
As to the second part, notice that Esakia spaces are Priestley spaces in which the downward-closure of a clopen set is a clopen set.
Therefore, we can define the map $\iota\colon \mathbb{B} \to \mathbb{A}$ by the assignment $b \mapsto \neg((\neg b){\downarrow})$. It can be readily verified that ι is the right adjoint of e and that moreover condition $a \to^{\mathbb{A}} b = \iota(\neg^{\mathbb{B}} e(a) \vee^{\mathbb{B}} e(b))$ holds.

Example 1: Gödel-Tarski translation

Remaining lemmas

- The BAE $(\mathbb{B}, \Box^{\mathbb{B}})$, with $\Box^{\mathbb{B}}$ defined above, is normal and is also an S4-modal algebra.
- Let $\mathbb{A}$, $\mathbb{B}$, $e \colon \mathbb{A} \hookrightarrow \mathbb{B}$ and $r \colon \mathbb{B}^X \to \mathbb{A}^X$ be as above. Then the Gödel-Tarski translation τ satisfies conditions (a) and (b) for any formula $\varphi \in \mathcal{L}_1$.
- Let $\mathbb{A}$ be a Heyting algebra and $\mathbb{B}$ a Boolean algebra such that $e \colon \mathbb{A} \hookrightarrow \mathbb{B}$ and $\iota \colon \mathbb{B} \to \mathbb{A}$ exist as above. Then for all intuitionistic formulas φ and ψ,

$$\mathbb{A} \models \varphi \leq \psi \quad \text{iff} \quad \mathbb{B} \models \tau(\varphi) \leq \tau(\psi),$$

  where τ is the Gödel-Tarski translation.

Example 2: co-Gödel-Tarski translation

- $\mathcal{L}_1$=co-intuitionistic propositional language with $\rightarrow$ replaced by $\succ\!\!-$
- $\mathcal{L}_2$=propositional S4 modal language with $\Diamond_{\geq}$
- $\mathbb{A}$: co-Heyting algebra
- $\mathbb{B}$: S4$\Diamond_{\geq}$ modal algebra constructed from $\mathbb{A}$
- $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$:order-embedding (to be proved)
- $c\colon \mathbb{B} \rightarrow \mathbb{A}$: the left adjoint of $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ (existence to be proved)
- $\Diamond_{\geq} : \mathbb{B} \rightarrow \mathbb{B}$ defined as $e \circ c$ ($\mathbb{B}$ being a normal S4$\Diamond_{\geq}$ modal algebra to be proved)
- translation σ defined as follows:
  $\sigma(p) = \Diamond_{\geq} p$
  $\sigma(\varphi \succ\!\!- \psi) = \Diamond_{\geq}(\neg\sigma(\varphi) \wedge \sigma(\psi))$
- $r : \mathbb{B}^X \rightarrow \mathbb{A}^X$: $U \mapsto c \circ U$
- condition (a) and (b) hold for r and σ (to be proved, by induction)

Example 2: co-Gödel-Tarski translation

Existence of Boolean algebra and left adjoint

For every co-Heyting algebra $\mathbb{A}$, there exists a Boolean algebra $\mathbb{B}$ such that $\mathbb{A}$ embeds into $\mathbb{B}$ via some order-embedding $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ which is a homomorphism of the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$, and has a left adjoint $c\colon \mathbb{B} \to \mathbb{A}$ verifying condition $a \succ\!\!-^{\mathbb{A}} b = c(\neg^{\mathbb{B}} e(a) \wedge^{\mathbb{B}} e(b))$. Finally, these facts lift to the canonical extensions of $\mathbb{A}$ and $\mathbb{B}$ as in the following diagram:

$$
\begin{tikzcd}
\mathbb{A}^\delta \arrow[r, hook, "e^\delta"] \arrow[r, phantom, shift right=3, "\top"] & \mathbb{B}^\delta \arrow[l, bend left, "c^\sigma"] \\
\mathbb{A} \arrow[u, hook] \arrow[r, hook, "e"] \arrow[r, phantom, shift right=3, "\top"] & \mathbb{B} \arrow[u, hook] \arrow[l, bend left, "c"]
\end{tikzcd}
$$

Example 2: co-Gödel-Tarski translation

Remaining lemmas

- The BAE $(\mathbb{B}, \Diamond^{\mathbb{B}})$, with $\Diamond^{\mathbb{B}}$ defined above, is normal and is also an $S4\Diamond$-modal algebra.
- Let $\mathbb{A}$, $\mathbb{B}$, $e \colon \mathbb{A} \hookrightarrow \mathbb{B}$ and $r \colon \mathbb{B}^X \to \mathbb{A}^X$ be as above. Then the co-Gödel-Tarski translation σ satisfies conditions (a) and (b) for any formula $\varphi \in \mathcal{L}_1$.
- Let $\mathbb{A}$ be a co-Heyting algebra and $\mathbb{B}$ a Boolean algebra such that $e \colon \mathbb{A} \hookrightarrow \mathbb{B}$ and $c \colon \mathbb{B} \to \mathbb{A}$ exist as above. Then for all co-intuitionistic formulas φ and ψ,

$$\mathbb{A} \models \varphi \leq \psi \quad \text{iff} \quad \mathbb{B} \models \sigma(\varphi) \leq \sigma(\psi),$$

where σ is the co-Gödel-Tarski translation.

Example 3: parametric Gödel-Tarski translation

- $\varepsilon : X \to \{1, \partial\}$: order-type of variables
- $\mathcal{L}_1$=bi-intuitionistic propositional language with both $\to$ and $\succ\!\!-$
- $\mathcal{L}_2$=propositional S4 modal language with both $\Box_{\leq}$ and $\Diamond_{\geq}$
- $\mathbb{A}$: bi-Heyting algebra
- $\mathbb{B}$: S4$\Box_{\leq}\Diamond_{\geq}$ modal algebra constructed from $\mathbb{A}$
- $e \colon \mathbb{A} \hookrightarrow \mathbb{B}$:order-embedding (to be proved)
- $c, \iota \colon \mathbb{B} \to \mathbb{A}$: the left,right adjoint of e (existence to be proved)
- $\Diamond_{\geq}, \Box_{\leq} : \mathbb{B} \to \mathbb{B}$ defined as $e \circ c$, $e \circ \iota$ ($\mathbb{B}$ being a normal S4$\Box_{\leq}\Diamond_{\geq}$ modal algebra to be proved)

Example 3: parametric Gödel-Tarski translation

- translation τ_ε defined as follows:

$$\tau_\varepsilon(p) = \begin{cases} \Box_{\leq} p & \text{if } \varepsilon(p) = 1 \\ \Diamond_{\geq} p & \text{if } \varepsilon(p) = \partial \end{cases}.$$

- $r_\varepsilon : \mathbb{B}^X \to \mathbb{A}^X$:

$$r_\varepsilon(U)(p) = \begin{cases} (\iota \circ U)(p) & \text{if } \varepsilon(p) = 1 \\ (c \circ U)(p) & \text{if } \varepsilon(p) = \partial \end{cases}$$

- condition (a) and (b) hold for r_ε and τ_ε (to be proved, by induction)
- Existence of Boolean algebra and left/right adjoints as well as the remaining lemmas are proved in a similar way.

Correspondence via translation

Preservation of Sahlqvist/inductive shape

For (co-, bi-) intuitionistic propositional language $\mathcal{L}$ (possibly with additional normal connectives), the following are equivalent for any order-type ε on X, and any $\mathcal{L}$-inequality $\varphi \leq \psi$:

- $\varphi \leq \psi$ is an (Ω, ε)-inductive $\mathcal{L}$-inequality;
- $\tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$ is an (Ω, ε)-inductive $\mathcal{L}^*$-inequality.

Example

In intuitionistic modal logic, take the formula $\Box\Diamond p \rightarrow \Diamond p$, it is ∂-Sahlqvist, so we take the parametric translation where $\varepsilon(p) = \partial$. The parametric Gödel-Tarski translation is $\Box_{\leq}(\Box\Diamond\Diamond_{\geq} p \rightarrow \Diamond\Diamond_{\geq} p)$.

Correspondence via translation

Main theorem

Every inductive $\mathcal{L}$-inequality has a first-order correspondent on $\mathcal{L}$-frames.

Proof sketch

$$
\begin{array}{ll}
 & \mathbb{F} \Vdash \varphi \leq \psi \\
\text{iff} & \mathbb{A} \models \varphi \leq \psi \\
\text{iff} & \mathbb{B} \models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi) \\
\text{iff} & \mathbb{F} \Vdash^* \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi) \\
\text{iff} & \mathbb{F} \models \mathsf{FO}(\varphi).
\end{array}
$$

Canonicity via translation

U-shaped argument

$$
\begin{array}{ccc}
\mathbb{A} \models \varphi \leq \psi & & \mathbb{A}^\delta \models \varphi \leq \psi \\
\Updownarrow & & \Updownarrow \\
\mathbb{B} \models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi) & \Leftrightarrow & \mathbb{B}^\delta \models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)
\end{array}
$$

Problems

- For parametrized translation, we cannot guarantee the existence of left (resp. right) adjoint for arbitrary (resp. co-) Heyting algebras. For distributive lattices, we can guarantee the existence of neither.
- One possible solution: consider the adjoint as maps going from $\mathbb{B}$ to $\mathbb{A}^\delta$ rather than $\mathbb{A}$, which guarantees the left arm to go through. (Notice that the interpretations of the box/diamond connectives are not mapping clopens to clopens anymore.)
- Problem again: the bottom line broke down due to lack of good topological properties of connectives.

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