An Ordering Condition for Groups

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Topology, Algebra, and Categories in Logic TACL 2017
Prague, June 27, 2017

Ordering Conditions

What do we mean by *ordering condition for groups?*

Given a class $\mathcal{G}$ of groups, it determines when a partial order of $\mathbf{G} \in \mathcal{G}$ can be extended to a (total) order of $\mathbf{G}$.

Examples

(Fuchs 1963)
Every partial order of a torsion-free abelian group $\mathbf{G}$ extends to an order of $\mathbf{G}$.

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A partial order of a group $\mathbf{G}$ extends to an order of $\mathbf{G}$ if, and only if, there is *a way* to extend it with any set of finitely many elements of G.

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Ordering Free Groups

Theorem (1)

Every free group can be totally ordered.

- Proved by Shimbireva (1947), Neumann (1948), Vinogradov (1949), and Bergman (1986).
- Non-trivial proofs that don't make use of Fuchs' ordering condition.

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We provide a new **algorithmic** ordering condition for groups.

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Invariant Orderings

Given a group $\mathbf{G} = \langle G, \cdot, ^{-1}, e \rangle$, a partial order $\leq$ of $\mathbf{G}$ is a partial order of G such that, for all $a, b, c, d \in G$,

$$a \leq b \implies cad \leq cbd.$$

- A partial order uniquely determines the set of its *strictly positive* elements, which is a normal subsemigroup that does not contain e.
- Normality follows from the fact that, for all $a, b \in G$:

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Given a group $\mathbf{G} = \langle G, \cdot, ^{-1}, e \rangle$, a partial order $\leq$ of $\mathbf{G}$ can be identified with a normal subsemigroup P of $\mathbf{G}$ omitting e:

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Given a group $\mathbf{G} = \langle G, \cdot, ^{-1}, e \rangle$, a (total) order $\leq$ of $\mathbf{G}$ can be identified with a normal subsemigroup P of $\mathbf{G}$ which does not contain e, and such that $P \cup P^{-1} \cup \{e\} = G$.

- A normal subsemigroup of G, omitting e, uniquely determines a partial order as the set of its strictly positive elements.
- Normality follows from the fact that, for all $a, b \in G$:

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Ordering Theorem

Let $\langle\langle S \rangle\rangle$ be the normal subsemigroup generated by $S \subseteq G$.

Theorem (Fuchs 1963)

The following are equivalent for a subset S of a group **G***:*

(1) *S extends to an order of* **G***.*

(2) *For all $a_1, \ldots, a_m \in G \setminus \{e\}$, there exist $\delta_1, \ldots, \delta_m \in \{-1, 1\}$ such that*

$$e \notin \langle\langle S \cup \{a_1^{\delta_1}, \ldots, a_m^{\delta_m}\} \rangle\rangle.$$

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An Algorithmic Condition

Given a finite subset $S \subseteq G$ of a group $\mathbf{G}$, we define $\vdash_{\mathbf{G}} S$ inductively by

(i) $\vdash_{\mathbf{G}} T \cup \{a, a^{-1}\}$,

(ii) $\vdash_{\mathbf{G}} T \cup \{ab\}$, if $\vdash_{\mathbf{G}} T \cup \{a\}$ and $\vdash_{\mathbf{G}} T \cup \{b\}$,

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(1) ⇒ (2)

By induction on the height of a **G**-derivation $\vdash_{\mathbf{G}} S$.

- It is clear that if $S = T \cup \{a, a^{-1}\}$, we have $e \in \langle\langle S \rangle\rangle$. ✓
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(2) $\Rightarrow$ (1): Key Lemma

Given a finite subset $S \cup \{c\} \subseteq G$ of a group $\mathbf{G}$, we call $\langle c, S \rangle$ a **finite pointed subset** of G. We define $\vdash^r_{\mathbf{G}} \langle c, S \rangle$, we define $\vdash_{\mathbf{G}} S$ inductively by

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By induction on $m \in \mathbb{N}$ for which there are $a_1, \ldots, a_m \in G \setminus \{e\}$ such that $e \in \langle\langle S \cup \{a_1^{\delta_1}, \ldots, a_m^{\delta_m}\}\rangle\rangle$, for all $\delta_1, \ldots, \delta_m \in \{-1, 1\}$.

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Main Result

Theorem

The following are equivalent for a finite subset S of a group $\mathbf{G}$:

(1) $\vdash_{\mathbf{G}} S$.

(2) *There exist $a_1, \ldots, a_m \in G \setminus \{e\}$ such that $e \in \langle\langle S \cup \{a_1^{\delta_1}, \ldots, a_m^{\delta_m}\}\rangle\rangle$, for all $\delta_1, \ldots, \delta_m \in \{-1, 1\}$.*

(3) *S does not extend to an order of $\mathbf{G}$.*

This Work

Results

- We establish a correspondence between the extension of partial orders on free groups and validity of equations in totally ordered groups.
- We give a new proof of Theorem (1) which makes use of Fuchs' ordering condition.

Lattice-ordered Groups

An ℓ-group $\mathbf{L} = \langle L, \wedge, \vee, \cdot, ^{-1}, e\rangle$ is an algebraic structure such that:

- $\langle L, \cdot, ^{-1}, e\rangle$ is a group;
- $\langle L, \wedge, \vee\rangle$ is a lattice, where the order

$$a \leq b \iff a \wedge b = a$$

  is a partial order of the group $\langle L, \cdot, ^{-1}, e\rangle$.

When $\leq$ is total, $\mathbf{L}$ is called a totally ordered group.

$\mathcal{OG}$ denotes the class of totally ordered groups, which generates the variety $\mathcal{RG}$ of representable ℓ-groups.

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Remark

By means of the strong distributivity properties of ℓ-groups, checking validity of equations in $\mathcal{RG}$ amounts to checking

$$\mathcal{RG} \models e \leq t_1 \vee \ldots \vee t_n,$$

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Free Groups

Let **F** be the **free group** over a non-empty set X of generators, its elements being reduced group terms obtained by cancelling occurrences of e, xx^{-1}, and $x^{-1}x$.

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Theorem

The following are equivalent for group terms $t_1, \ldots, t_n$:

(1) $\vdash_{\mathbf{F}} \{t_1, \ldots, t_n\}$.

(2) *There exist $s_1, \ldots, s_m \in F \setminus \{e\}$ such that* $e \in \langle\langle\{t_1, \ldots, t_n, s_1^{\delta_1}, \ldots, s_m^{\delta_m}\}\rangle\rangle$, *for all $\delta_1, \ldots, \delta_m \in \{-1, 1\}$.*

(3) $\{t_1, \ldots, t_n\}$ *does not extend to an order of* $\mathbf{F}$.

(4) $\mathcal{RG} \models e \leq t_1 \vee \ldots \vee t_n$.

Free Groups

Theorem

The following are equivalent for group terms $t_1, \ldots, t_n$:

(1) $\vdash_{\mathbf{F}} \{t_1, \ldots, t_n\}$.

(2) *There exist $s_1, \ldots, s_m \in F \setminus \{e\}$ such that $e \in \langle\langle\{t_1, \ldots, t_n, s_1^{\delta_1}, \ldots, s_m^{\delta_m}\}\rangle\rangle$, for all $\delta_1, \ldots, \delta_m \in \{-1, 1\}$.*

(3) $\{t_1, \ldots, t_n\}$ *does not extend to an order of* $\mathbf{F}$.

(4) $\mathcal{RG} \models e \leq t_1 \vee \ldots \vee t_n$.

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$\Uparrow$ *By induction on the height of* $\vdash_{\mathbf{F}} \{t_1, \ldots, t_n\}$

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Ordering Free Groups

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Every free group can be totally ordered.

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(3) x *does extend to an order of* $\mathbf{F}$.

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Concluding Remarks

- Can this approach provide new insights into the problem of deciding validity of equations in totally ordered groups?
- Can we use our conditions to prove ordering results for other groups (e.g., fundamental groups of surfaces)?
- This is not an isolated case (e.g., variety of ℓ-groups, group varieties of representable ℓ-groups). How far can we get? (e.g., normal-valued ℓ-groups)

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Thank you!