Multiplicative derivations of commutative residuated lattices

TACL 2017

26-30 June 2017, Prague

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Ring Theory

derivation $f : A \to A$

$$f(x + y) = f(x) + f(y)$$

$$f(x \cdot y) = f(x) \cdot y + x \cdot f(y)$$

$$\Downarrow$$

Residuated Lattice Theory

multiplicative derivation $d : L \to L$

$$d(x \wedge y) = (dx \odot y) \vee (x \odot dy) \qquad (\forall x, y \in L)$$

Results

(1) $\mathsf{Fix}_d(L)$ **is a residuated lattice and**

$$L/\mathsf{ker}(d) \cong \mathsf{Fix}_d(L);$$

(2) **A map** $(d/F)(x/F) = dx/F$ **is a good ideal derivation of** L/F**;**

(3) **The quotient residuated lattices** $\mathsf{Fix}_{d/F}(L/F)$ **and** $\mathsf{Fix}_d(L)/d(F)$ **are isomorphic,**

$$\mathsf{Fix}_{d/F}(L/F) \cong \mathsf{Fix}_d(L)/d(F).$$

$\boxed{\textbf{Definition}}$ $(L, \wedge, \vee, \odot, \to, 0, 1)$ **is called a** *residuated lattice* **if**

(1) $(L, \wedge, \vee, 0, 1)$ **is a bounded lattice;**

(2) $(L, \odot, 1)$ **is a commutative monoid with unit element 1;**

(3) **For all** $x, y, z \in L$**,**

$$x \odot y \leq z \iff x \leq y \to z.$$

$\boxed{\textbf{Proposition 1}}$ **For all** $x, y, z \in L$**, we have**

(1) $0' = 1, 1' = 0$;

(2) $x \odot x' = 0$;

(3) $x \leq y \;\Leftrightarrow\; x \to y = 1$;

(4) $x \odot (x \to y) \leq y$;

(5) $x \leq y \;\Rightarrow\; x \odot z \leq y \odot z, z \to x \leq z \to y,$

$$y \to z \leq x \to z;$$

(6) $1 \to x = x$;

(7) $(x \vee y) \odot z = (x \odot z) \vee (y \odot z)$;

(8) $(x \vee y)' = x' \wedge y'$;

(9) $(x'' \odot y'')'' = (x \odot y'')'' = (x \odot y)''$.

$\boxed{\textbf{Definition}}$ $x \in L$ **is called** *complemented* **if**

$$\exists y \in L \text{ s.t. } \quad x \wedge y = 0 \text{ and } x \vee y = 1.$$

By $B(L)$**, we mean the set of all complemented elements,**

$$B(L) = \{x \in L \,|\, \exists y \in L \text{ s.t. } x \wedge y = 0, x \vee y = 1\}.$$

$\boxed{\textbf{Proposition 2}}$ **For a residuated lattice L,**

(1) $x \in B(L)$ **if and only if** $x \vee x' = 1$,
where $x' = x \to 0$;

(2) **If** $x \in B(L)$ **then** $x \wedge y = x \odot y$ **for all** $y \in L$;

(3) **If** $x \oplus y = x \vee y$ **for all** $y \in L$ **then** $x \vee x' = 1$,
where $x \oplus y = (x' \odot y')'$;

(4) $B(L)$ **is a Boolean subalgebra of** L.

$\boxed{\textbf{Definition}}$ $d : L \to L$ **is called a** *multiplicative derivation* **(or simply** *derivation***) of** L **if**

$$d(x \wedge y) = (dx \odot y) \vee (x \odot dy) \qquad (\forall x, y \in L)$$

Derivation d **is called**

monotone **: if** $x \leq y \Rightarrow dx \leq dy$**.**

ideal **: if monotone and** *contractive***,i.e.,** $dx \leq x$**.**

good **: if** $d1 \in B(L)$**.**

$\boxed{\text{Example 1}}$ $X = \{0, a, 1\}$ $(0 < a < 1)$ **is a residuated lattice by** $x \wedge y = x \odot y = \min\{x, y\}, x \vee y = \max\{x, y\}$ **and**

$$x \to y = \begin{cases} 1 & (\text{if } x \leq y) \\ y & (\text{otherwise}). \end{cases}$$

A map d_a **defined by** $d_a(x) = x \wedge a$ **is a** *contractive monotone* **derivation, but** *not* *good*.

Because, $d_a(1) = a \notin B(X)$ **for** $B(X) = \{0, 1\}$.

$\boxed{\text{Example 2}}$ **As another example,**

a map $f : X \to X$ **defined by** $f1 = 0 = f0, fa = a$ **is**

contractive **and**

good **(**$f(1) = 0 \in B(X) = \{0, 1\}$**), but**

not *monotone* **(**$a \leq 1$ **but** $f(a) \not\leq f(1)$**).**

Proposition 3 **Let d be a derivation of L.**

For all $x, y \in L$,

(1) $d0 = 0$;

(2) $dx \geq x \odot d1$;

(3) $dx^n = x^{n-1} \odot dx$ **for all** $n \geq 1$;

(4) $x \odot y = 0 \;\Rightarrow\; dx \odot y = x \odot dy = dx \odot dy = 0$;

(5) $d(x') \leq (dx)'$.

$\boxed{\text{Theorem 4}}$ **[P.He, X.Xin and J.Zhan, 2016]**

For a good derivation d, the following are equivalent:

(1) d **is an ideal derivation;**

(2) $dx \leq d1$**;**

(3) $dx = x \odot d1 = x \wedge d1$**;**

(4) $d(x \wedge y) = dx \wedge dy$**;**

(5) $d(x \vee y) = dx \vee dy$**;**

(6) $d(x \odot y) = dx \odot dy$**.**

$\boxed{\boxed{\textbf{Fact}}}$ $d = d^2$ **for a good ideal derivation** d**.**

If d **is a good ideal derivation, then we have** $dx = x \wedge d1$ **and thus** $d^2x = d(dx) = d(x \wedge d1) = (x \wedge d1) \wedge d1 = x \wedge d1 = dx$**. This means that** $d^2 = d$**.**

For a derivation d **of** L**, we denote by** $\mathsf{Fix}_d(L)$ **the set of all fixed elements for** d**,**

$$\mathsf{Fix}_d(L) = \{x \in L \mid dx = x\}.$$

$\boxed{\text{Proposition 5}}$ **For a good ideal derivation d of a residuated lattice L, we have $\mathrm{Fix}_d(L) = d(L)$.**

$\boxed{\text{Lemma 6}}$ **Let d be a good ideal derivation. Then we have $d(dx \to dy) = d(x \to y)$ for all $x, y \in L$.**

We define some operations in $\mathrm{Fix}_d(L) = d(L)$ by

$$dx \wedge dy = d(x \wedge y);$$

$$dx \vee dy = d(x \vee y);$$

$$dx \odot dy = d(x \odot y);$$

$$dx \mapsto dy = d(dx \to dy).$$

$\boxed{\text{Theorem 7}}$ **Let L be a residuated lattice and d be a good ideal derivation of L. Then,** $\mathsf{Fix}_d(L) = (\mathsf{Fix}_d(L), \wedge, \vee, \odot, \mapsto, 0, d1)$ **is a residuated lattice.**

However, it is not a subalgebra of L in general.

$$\Downarrow$$

$\mathsf{Fix}_d(L)$ **is a subalgebra of L** $\Leftrightarrow \quad d = id_L$

Moreover, we see that a good ideal derivation d is a homomorphism from L to $\mathsf{Fix}_d(L)$.

$$\Downarrow$$

$\boxed{\textbf{Theorem 8}}$ **For every good ideal derivation d, it is a homomorphism from L to $\mathsf{Fix}_d(L)$ and hence**

$$L/\,\mathsf{ker}(d) \cong \mathsf{Fix}_d(L),$$

where $\mathsf{ker}(d) = \{(x, y) \in L \times L \,|\, dx = dy\}$**.**

Definition **filter and d-filter**

A non-empty subset F is called a *filter* if

(F1) If $x, y \in F$ then $x \odot y \in F$;

(F2) If $x \in F$ and $x \leq y$ then $y \in F$.

For a good ideal derivation d, a filter F is called a d-*filter* if $x \in F$ implies $dx \in F$ for all $x \in L$.

$$\mathcal{F}(L) = \{F \,|\, F \text{ is a filter}\}$$

$$\mathcal{F}_d(L) = \{F \,|\, F \text{ is a } d\text{–filter}\}$$

By $[S)$ (or $[S)_d$), we mean the generated filter (or generated d-filter, respectively) by S.

$\boxed{\textbf{Proposition 9}}$ **[Characterization of d-filters]**

For a good ideal derivation d and a non-empty subset S, we have $[S)_d = [S \cup d(S))$.

Corollary $\mathcal{F}_d(L)$ **is a complete Heyting algebra.**

Corollary **If** F **is a filter, then** $[F)_d = [d(F))$.

Corollary **For any** $F \in \mathcal{F}(L)$, F **is a** d**–filter if and only if** $F = [d(F))$.

$\boxed{\textbf{Proposition 10}}$ **Let d be a good ideal derivation of L. Then we have**

(1) $F \in \mathcal{F}_d(L) \Rightarrow d(F) \in \mathcal{F}(d(L))$**;**

(2) $G \in \mathcal{F}(d(L)) \Rightarrow d^{-1}(G) \in \mathcal{F}_d(L)$**,**

where $d^{-1}(G) = \{x \in L \,|\, dx \in G\}$**.**

$\boxed{\textbf{Proposition 11}}$ **Let d be a good ideal derivation and $F \in \mathcal{F}_d(L)$. A map $d/F : L/F \to L/F$ defined by $(d/F)(x/F) = dx/F$ is a good ideal derivation of L/F.**

From the above, the quotient structure

$$(d/F)(L/F) = (\mathsf{Fix}_{d/F}(L/F), \wedge, \vee, \odot, \mapsto, 0/F, 1/F)$$

is a residuated lattice for any good ideal derivation d and d-filter F.

Moreover, since F is the d-filter, $d(F)$ is the filter of $d(L)$ and thus the quotient structure $d(L)/d(F)$ forms a residuated lattice.

$\boxed{\textbf{Question}}$ **What is the relation between two residuated lattices $(d/F)(L/F)$ and $d(L)/d(F)$?**

$\boxed{\textbf{Lemma 12}}$

If $F \in \mathcal{F}_d(L)$ then we have $F \cap d(L) = d(F)$.

$$\Downarrow$$

$\boxed{\textbf{Theorem 13}}$ **Let d be a good ideal derivation and F be a d-filter of L.**

Then $(d/F)(L/F) = \mathrm{Fix}_{d/F}(L/F)$ is isomorphic to $d(L)/d(F)$, that is,

$$(d/F)(L/F) = \mathrm{Fix}_{d/F}(L/F) \cong d(L)/d(F).$$

Thank you for your attention!